BROWNIAN MOTION

Reference :- Peter Mörters and Yuval Peres
"Brownian Motion"

Overview :- → Brownian motion is a random continuous f^n on $[0,\infty)$ s.t.

i) $B(0) = 0$

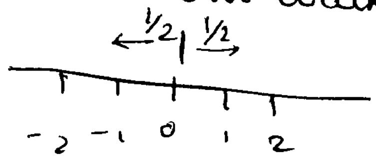
ii) $t_1 < s_1 < t_2 < s_2 \cdots t_k < s_k$

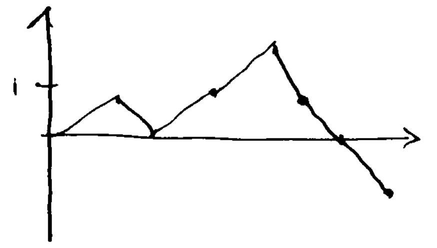
then $B_{s_k} - B_{t_k}$ independent $N(0, s_k - t_k)$

iii) $t \to B_t$ is continuous.

→ Weiner, on the space $C[0,\infty)$ defined a measure and called it as Weiner measure.

→ Random walk :-





Q: P{returning to 0 infinitely many times}

$$= \begin{cases} 1 & \text{in } \mathbb{Z}, \mathbb{Z}^2 \\ 0 & \text{in } \mathbb{Z}^3, \mathbb{Z}^4 \end{cases}$$

In case of BM, it will move either ε fwd or bkd where ε is really small.

Q: $t \to B_t$ is cts. How cts?

An : It is nowhere differentiable

→ Wierstrass function

B_t is Hölder continuous of order $< \frac{1}{2}$.

Local maxima/minima is dense in any interval.

Q: Let $X_1, X_2, \ldots$ be iid ± 1 w.p. $\frac{1}{2}$

Let $S_n = X_1 + \cdots X_n$

By LLN, $\frac{S_n}{n} \to 0$ a.s.

& CLT, $\frac{S_n}{\sqrt{n}} \xrightarrow{d} N(0,1)$

Note that $\frac{S_n}{n^\alpha} \to 0$ $\frac{1}{2} < \alpha < 1$.

LIL:

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{n \log\log n}} = 2 \quad \text{a.s}$$

1 in expansion of π behaves like random sequence.

$n \geq 1$

$$\mu(n) = \begin{cases} (-1)^k & \text{if } n = p_1 p_2 \cdots p_K, \ p_i\text{'s are distinct} \\ 0 & \text{o.w.} \end{cases}$$

$\therefore \mu(4) = 0$ $\qquad M(x) = \sum_{n \leq x} \mu(n)$

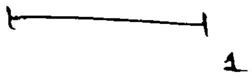
$\mu(10) = 1$ $\qquad$ order of growth: $M(x)$

$\mu(70) = -1$

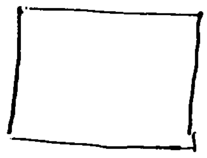
conjecture: $\lim_{x\to\infty} \frac{M(x)}{x^{1/2+\varepsilon}} = 0$.

Fractals from B.M.

$A \subseteq \mathbb{R}^d$ compact — cover A with disks of diameter ε



$\rightarrow \frac{1}{\varepsilon}$



$\rightarrow \frac{1}{\varepsilon^2}$

Clearly for 3-D $\rightarrow \frac{1}{\varepsilon^3}$ ... so on.

Let N_ε = minimal # of balls of diameter ε needed to cover A

Define $\dim_M(A) = \lim_{\varepsilon \downarrow 0} \frac{\log N_\varepsilon}{\log(\varepsilon^{-1})}$ if it exists.

This is called as Minkowski's dimension.

$\frac{1}{3}$ - Cantor set is a fractal with dimension $\frac{\log 2}{\log 3}$.

There are example where the limit doesnot exist.

In case of deterministic fn in $\mathbb{R}^1$, the graph is normally 1 if it is differentiable. But in case of Brownian motion $\dim(\text{Graph} = \{(t, B_t) / t \in [0,1]\})$ $= 3/2$ a.s.

In case d dimensional B.M. again is a fractal object has dimension is 2 $\forall\, d \geq 2$.

1-d B.M : Zero set of BM has dim $= \frac{1}{2}$ a.s.

What sets does ^a 3-dim BM hit?

$B_t \sim N_3(0, t I_3)$

$P(B$ hits a sphere$) > 0$

$P(B$ hits a ~~g~~ pt$) = 0$

$P(B$ hits a surface$) \iff$ U can distribute unit charge over A so that the T.E. is finite. (*)

Coulomb's Law $q_1 \xrightarrow{r} q_2$. $F \propto \frac{q_1 q_2}{r^2}$

• q : Energy due to the charge is q/r

In case of discrete distribution of charge the energy is infinite else it can be finite.

(*) A is not the set where B.M. starts.

6-8-09

1) Gaussian (Normal) random variables–

• $(\Omega, \mathcal{F}, P)$ be a probability space

X: a r.v on Ω if $X : \Omega \to \mathbb{R}$ measurable.

$X \sim N(0,1)$ if
$$P[X \in [a,b]] = \int_a^b \frac{e^{-t^2/2}}{\sqrt{2\pi}} dt.$$

$Y = \sigma X + \mu \sim N(\mu, \sigma^2)$ $\sigma > 0$, $\mu \in \mathbb{R}$

Let $X_1, \dots X_n$ be independent $N(0,1)$

Let $\underline{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}$

$Y = A\underline{X} + \mathcal{M}$ $\quad A_{m\times n}$ matrix $\quad M_{m\times 1}$

$\sim N_m(M, \Sigma)$ $\quad$ where $\Sigma = AA^t$

If $Y \sim N_m(\mu, \Sigma)$ $\quad EY_i = \mu_i$, $\operatorname{cov}(Y_i, Y_j) = \sigma_{ij}$

where $\Sigma = ((\sigma_{ij}))_{i,j \le m}$.

Note :- 1) If $\sigma_{ij} = 0$ $\forall i \neq j$, then $X_1, \cdots X_n$ are indep.

$X_j \sim N(\mu_j, \sigma_{jj})$

2) If Σ is non singular, then $N(\mu, \Sigma)$ has density

$$\frac{e^{-(\underline{x}-\mu)^t \Sigma^{-1}(x-\mu)/2}}{(2\pi)^{n/2} |\Sigma|^{1/2}}$$

w.r.t. Lebesgue on $\mathbb{R}^n$

Fact 1: Let $X \sim N(0,1)$ $\quad E(X) = 0$, $E(X^2) = 1$

$E[X^{2n+1}] = 0$

$E[X^{2n}] = (2n-1)(2n-3) \cdots 1$

$=$ # of matchings of the set $1, 2, \cdots 2n$

Ex: $X \sim N_m(\underline{0}, \Sigma)$

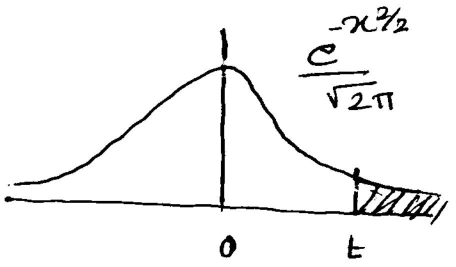
$$E[X_1 X_2 \cdots X_m] = \sum_{\substack{M:\text{ matching} \\ \text{of } 1,2\cdots m}} \omega(M)$$

Note for m odd it will be zero.

where $\omega(M) = \prod_{\{i, n_i\}} \sigma_{i n_i}$

Fact 2: $t > 0$

$$P\{X > t\} = \int_t^\infty \frac{e^{-u^2/2}}{\sqrt{2\pi}}\,du$$



$$\leq \frac{1}{\sqrt{2\pi}\,t}\int_t^\infty \frac{u\cdot e^{-u^2/2}}{\sqrt{2\pi}}\,du$$

$$= \frac{e^{-t^2/2}}{\sqrt{2\pi}\cdot t}$$

Ex*: Show that $\exists\ c > 0$ s.t. $\forall\ t > 1$ $P\{X > t\} \geq c\,\frac{e^{-t^2/2}}{t}$

Fact 3 Suppose $\mu_n \to \mu$, $\sigma_n^2 \to \sigma^2$ then

$N(\mu_n, \sigma_n^2) \xrightarrow{d} N(\mu, \sigma^2)$

2> Brownian motion: $(\Omega, \mathcal{F}, P)$ - a probability space

$X = (X_t)_{t \geq 0}$ a collection of r.v.s on Ω

we say that X is a std B.M if

(i) $X_0 = 0$

(ii) For any $0 < t_1 < t_2 \cdots < t_n$, the r.v.s

$X_{t_1}, X_{t_2} - X_{t_1}, \ldots\ldots X_{t_n} - X_{t_{n-1}}$ are indep

and $X_t - X_s \sim N(0, t-s)$ $\forall\ s < t$.

(iii) For a.e $\omega \in \Omega$,

the f^n $t \longrightarrow X_t(\omega)$ is cts on $[0, \infty)$

Q: i) Exits? (ii) Unique?

3) The measure space $C[0,\infty)$ and Wiener measure:

Start with $C[0,1] = \{ f : [0,1] \to \mathbb{R} \mid f \text{ cts} \}$.

Metric: $d(f,g) = \| f-g \|_{sup}$

$$= \sup_{x \in [0,1]} |f(x) - g(x)|$$

Let $\mathcal{B}_{[0,1]}$ = smallest σ-field containing all open sets in $C[0,1]$.

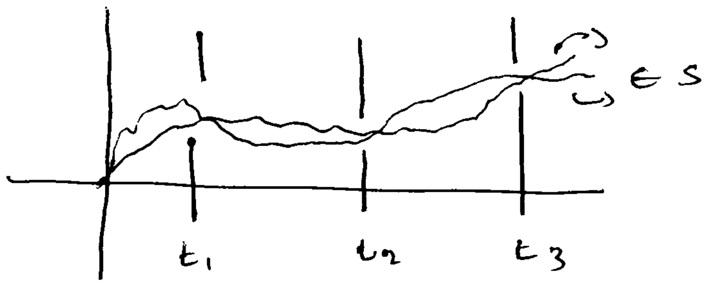
(Borel σ-field on $C[0,1]$).

Take any $0 \leq t_1 < \cdots t_n \leq 1$ and B - a borel set in $\mathbb{R}^n$

Let $S = \{ f \in C[0,1] \mid (f(t_1), \ldots, f(t_n)) \in B \}$

then S is called as finite dimensional cylinder.

eg: $B = (a_1, b_1) \times (a_2, b_2) \times (a_3, b_3)$



Ex: The set of all f. d. cylinders generate $\mathcal{B}_{[0,1]}$

For $C[0,\infty)$

$$d(f,g) := \sum_{T=1}^{\infty} \frac{1}{2^T} \frac{\| f-g \|_{C[0,T]}}{1 + \| f-g \|_{C[0,T]}}$$

<u>Def^n</u> :— A measure μ on $(C[0,\infty), \mathcal{B}_{[0,\infty)})$ is called as the Wiener measure if

(i) $\mu\{f \mid f(0) = 0\} = 1$

(ii) $\mu\{f \mid f(t_i) \in (a_i, b_i),\ i = 1, 2 \cdots k\}$ $0 \le t_1 < \cdots < t_k < \infty$

$$= \int_{a_1}^{b_1}\int_{a_2}^{b_2} \cdots \int_{a_k}^{b_k} \frac{e^{-u_1^2/2t_1}}{\sqrt{2\pi t_1}} \cdot \frac{e^{-\frac{(u_2-u_1)^2}{2(t_2-t_1)}}}{\sqrt{2\pi(t_2-t_1)}} \cdots \frac{e^{-\frac{(u_k-u_{k-1})^2}{2(t_k-t_{k-1})}}}{\sqrt{2\pi(t_k-t_{k-1})}}\, du_1 \cdots du_k.$$

Q: Does such μ exists?

<u>Remark</u>: (1) Suppose μ exists. Then take $\Omega = C[0,\infty)$, $\mathcal{F} = \mathcal{B}_{[0,\infty)}$, $P = \mu$.

Define $X_t(\omega) = \omega(t)$ for $t \ge 0$

Then $(X_t)_{t\ge 0}$ is a std B.M.

(2) Suppose $(\Omega, \mathcal{F}, P)$ is some pr. space and $X = (X_t)_{t\ge 0}$ is std B.M. ($\exists\ E \in \mathcal{F}$, $P(E) = 0$ and for $\omega \in \Omega\setminus E$, $t \to X_t(\omega)$ is continuous.)

Then Let $\mu = PX^{-1}$ where

$X: \Omega \to C[0,\infty)$ &

For $\omega \in \Omega$

$$X(\omega) = \begin{cases} \{t \to X_t(\omega)\} & \text{if } \omega \in E^c \\ 0 & \text{if } \omega \in E \end{cases}$$

Then μ is a Wiener measure.

4) Levy's construction of B. M. :—

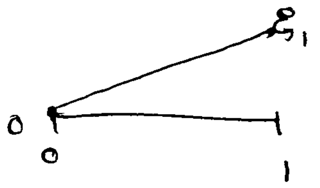
* Let $(\Omega, \mathcal{F}, P)$ be an prob space with $\xi_1, \xi_2 \dots \xi_n \dots$ iid $N(0,1)$.

(Note: This need not exist in all p-space. For e.g. $\Omega = \{0,1\}$).

Eg: ([0,1], Borel σ-field, Lebesgue measure).
In this space it exists.

* First we construct B, M on $[0,1]$

$(X_t)_{t \in [0,1]}$



$$F_0(t) = \begin{cases} 0, & t = 0 \\ \xi_1 & t = 1 \\ \text{linear in between.} \end{cases}$$

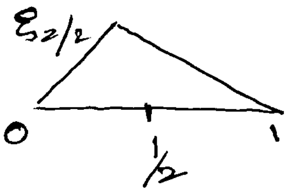
$B_0 = F_0$

$B_0(1) - B_0(0) = N(0,1)$

Note $B_0(\tfrac{1}{2}) - B_0(0) = \frac{\xi_1}{2}$

$B_0(1) - B_0(\tfrac{1}{2}) = \frac{\xi_1}{2}$

→ neither indep nor $N(0, \tfrac{1}{2})$

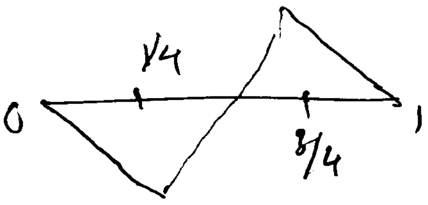


$$F_1(t) = \begin{cases} 0 & t = 0, 1 \\ \xi_2/2 & t = 1/2 \\ \text{linear in bet}^n. \end{cases}$$

$B_1 = B_0 + F_1 = F_0 + F_1$

$$\left.\begin{aligned} B_1(\tfrac{1}{2}) - B_1(0) &= \frac{\xi_1}{2} + \frac{\xi_2}{2} \\ B_1(1) - B_1(\tfrac{1}{2}) &= \frac{\xi_1}{2} - \frac{\xi_2}{2} \end{aligned}\right\} \text{indep \& } N(0, \tfrac{1}{2})$$

$$F_2(t) = \begin{cases} 0 & \text{for } t = 0, \tfrac{1}{2}, 1 \\ \xi_3/a_1 & t = 1/4 \\ \xi_4/a_2 & t = \tfrac{3}{4} \\ \text{linear in bet} \end{cases}$$



$\therefore\ B_2 := F_0 + F_1 + F_2$

$$B_2(1/4) - B_2(0) = \frac{\xi_1}{4} + \frac{\xi_2}{4} + \frac{\xi_3}{a_1}$$

This variable has variance $= \frac{1}{16} + \frac{1}{16} + \frac{1}{a_1^2} = \frac{1}{4}$

$\therefore\ a_1 = \sqrt{8}$

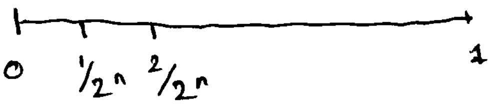
Ily $\quad B_2(1/2) - B_2(1/4) = \frac{\xi_1}{4} + \frac{\xi_2}{4} - \frac{\xi_3}{\sqrt{8}}$

(Note calculating a_2 is same as that of a_1)

Clearly we get independence & N condⁿ.

11-8-09

At n^{th} stage



$$F_n(t) = \begin{cases} 0 & \text{if } t = \frac{2k}{2^n},\ k = 0, 1, \cdots 2^{n-1} \\ \frac{\xi_{2^{n-1}+k}}{\sqrt{2^{(n+1)}}} & t = \frac{2k-1}{2^n},\ k = 1, \cdots 2^{n-1} \\ \text{linear in bet}^n. & \end{cases}$$

Set $B_n(t) = F_0(t) + F_1(t) + \cdots F_n(t)$.

1) Estimate $\|F_n\|_{sup}$.

$$\|F_n\|_{sup} = \left(\max_{2^{n-1} \le k \le 2^n} |\xi_k|\right) \frac{1}{\sqrt{2^{n+1}}}$$

$$P\{\|F_n\|_{sup} > x\} = P\{\max_{2^{n-1} \le k \le 2^n} |\xi_k| \geq x\sqrt{2^{n+1}}\}$$

$$\leq 2^{n-1} P\{|\xi_1| \geq x\sqrt{2^{n+1}}\}$$

(Note $P\{|\xi_1| > t\} \leq \frac{2e^{-t^2/2}}{t}$, $t > 0$)

$$\leq 2^{n-1} \cdot \frac{2 e^{-x^2 2^n}}{x\sqrt{2^{n+1}}}$$

(from tail bound for Gaussian)

$$= \frac{e^{-[x^2 2^n] + [n \log 2]}}{x\sqrt{2^{n+1}}}$$

Take $x^2 2^n - n \log 2 >> 0$

$$x >> \frac{\sqrt{n \log 2}}{2^{n/2}}$$

$\therefore$ let $x_n = \dfrac{c\sqrt{n}}{2^{n/2}}$, $c > \sqrt{\log 2}$

For this choice of x_n

$$P\left\{ \|F_n\|_{sup} > \frac{c\sqrt{n}}{2^{n/2}} \right\} \leq \frac{2^n e^{-cn \log 2}}{\sqrt{2}\sqrt{n \log 2}}$$

$$\leq 2^n e^{-cn\log 2}$$

Hence $\sum_n P\left\{ \|F_n\|_{sup} > \dfrac{c\sqrt{n}}{2^{n/2}} \right\} < \infty$

(if $c > \sqrt{\log 2}$)

$\therefore$ Using Borel Cantelli lemma we get

w.p.1 $\exists\, N < \infty$ (random) s.t. $\forall n \geq N$

$$\|F_n\|_{sup} \leq \frac{c\sqrt{n}}{2^{n/2}}.$$

In particular, w.p.1

$$\sum_n \|F_n\|_{sup} < \infty$$

2) Set $B(t) = \lim_{n\to\infty} B_n(t)$

$$= \sum_{n=0}^{\infty} F_n(t)$$

(By step 1, w.p.1, the series cgs uniformly.)

Hence for ω in this set of prob 1 $t \mapsto B(t)$ is well defined and cts.

3) $(B_t)_{0\le t\le 1}$ is B.M. (run for unit time)

(i) $t \to B(t)$ is continuous (for a.e $\omega \in \Omega$).

(ii) $B(0) = 0$ a.s. ($\because F_n(0) = 0, \forall n$)

(iii) Independent increments?

Take $0 \le t_1 < t_2 \le 1$

First assume, t_1, t_2 are dyadic rationals

i.e $t_1 = \frac{k}{2^n}$, $t_2 = \frac{l}{2^n}$ for some $n \ge 1$ & $k \le l \le 2^n$

Then $F_j(t_1) = 0$ $F_j(t_2) = 0$ for $j \ge (n+1)$

Hence $B(t_1) = B_n(t_1)$

$B(t_2) = B_n(t_2)$

Claim: $B_n(1/2^n), B_n(2/2^n) - B_n(1/2^n)$, $B_n(3/2^n) - B_n(2/2^n), \cdots B_n(1) - B_n\left(\frac{2^n-1}{2^n}\right)$

are iid $N(0, 1/2^n)$

For a moment, let us assume this claim to be true, we have.

$$B_n\left(\frac{k}{2^n}\right) = B_n\left(\frac{1}{2^n}\right) + \left(B_n\left(\frac{2}{2^n}\right) - B_n\left(\frac{1}{2^n}\right)\right) + \cdots$$
$$\downarrow$$
$$B(t_1) \qquad + \left(B_n\left(\frac{k}{2^n}\right) - B_n\left(\frac{k-1}{2^n}\right)\right)$$

$$B(t_2) - B(t_1)$$
$$= \left(B_n\left(\frac{k+1}{2^n}\right) - B_n\left(\frac{k}{2^n}\right)\right) + \cdots$$
$$+ \left(B_n\left(\frac{l}{2^n}\right) - B_n\left(\frac{l-1}{2^n}\right)\right)$$

Hence by the claim

$$B(t_1) \sim N\left(0, \frac{k}{2^n}\right) = N(0, t_1)$$

$$B(t_2) - B(t_1) \sim N\left(0, \frac{l-k}{2^n}\right) = N(0, t_2 - t_1)$$

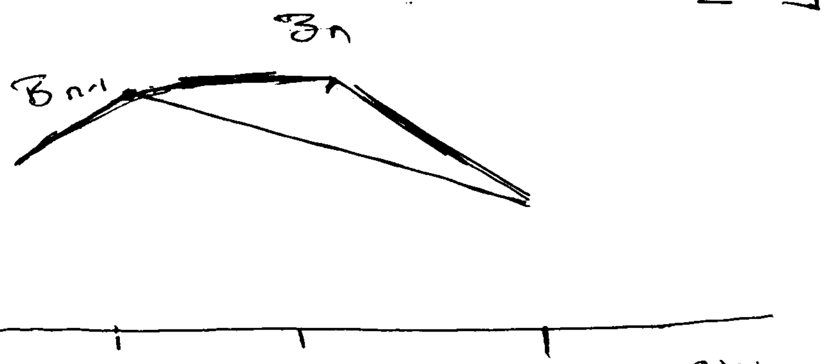
& they are independent.

<u>Proof</u> of <u>the claim</u>: By induction.

* (checked for $n = 0, 1, 2 \ldots$)
* Assume this holds up to $n-1 \Rightarrow \left\{ \left[B_{n-1}\left(\frac{j}{2^{n-1}}\right) - B_{n-1}\left(\frac{j-1}{2^{n-1}}\right)\right]_{1 \le j \le 2^{n-1}} \text{ are iid } N\left(0, \frac{1}{2^{n-1}}\right) \right.$

$$B_n = B_{n-1} + F_n$$

Consider an interval $\left[\frac{k}{2^{n-1}}, \frac{k+1}{2^{n-1}}\right]$



$$\frac{2k}{2^n} = \frac{k}{2^{n-1}} \qquad \frac{2k+1}{2^n} \qquad \frac{k+1}{2^{n-1}} = \frac{2k+2}{2^n}$$

$$F_n\left(\frac{2k+1}{2^n}\right) = \frac{X}{\sqrt{2^{n+1}}}$$

$$X \sim N(0,1) \quad (X = \xi_{2^{n-1}+k})$$

$$F_n\left(\frac{2k}{2^n}\right) = 0 = F_n\left(\frac{2k+2}{2^n}\right)$$

Hence $B_n\left(\frac{2k+1}{2^n}\right) - B_n\left(\frac{2k}{2^n}\right)$

$$= \frac{1}{2}\left[B_{n-1}\left(\frac{k+1}{2^{n-1}}\right) - B_{n-1}\left(\frac{k}{2^{n-1}}\right)\right] + \underbrace{\frac{X}{\sqrt{2^{n+1}}}}_{= \frac{\xi_{2^{n-1}+k}}{\sqrt{2^{n+1}}}}$$

$$B_n\left(\frac{2k+2}{2^n}\right) - B_n\left(\frac{2k+1}{2^n}\right)$$

$$= \frac{1}{2}\left[B_{n-1}\left(\frac{k+1}{2^{n-1}}\right) - B_n\left(\frac{k}{2^{n-1}}\right)\right]$$

$$- \frac{\xi_{2^{n-1}+k}}{\sqrt{2^{n+1}}}.$$

Hence we see that

$$\left\{ B_n\left(\frac{j}{2^n}\right) - B_n\left(\frac{j-1}{2^n}\right) \right\}_{1 \le j \le 2^n}$$

are iid $N(0, \frac{1}{2^n})$.

$\blacksquare$

* Need to check independent increments for general $0 \le t_1 < t_2 \le 1$

$\exists$ dyadic rationals, $t_{1,n} \to t_1$, $t_{2,n} \to t_2$ (actually $t_{1n} < t_{2n}$).

$B(t_{1n})$, $B(t_{2n} - B(t_{1n}))$ are independent $N(0, t_{1n})$, $N(0, (t_{2n} - t_{1n}))$ resp.

By continuity of B,

$$B(t_{1n}),\ B(t_{2n})-B(t_{1n}) \xrightarrow{a.s} (B(t_1),\ B(t_2)-B(t_1))$$

$$\therefore N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} t_{1n} & 0 \\ 0 & t_{2n}-t_{1n} \end{bmatrix} \right) \xrightarrow{d} N\left(\underline{0}, \begin{bmatrix} t_1 & 0 \\ 0 & t_2-t_1 \end{bmatrix} \right)$$

$$\therefore (B(t_1),\ B(t_2)-B(t_1)) \sim N\left(\underline{0}, \begin{bmatrix} t_1 & 0 \\ 0 & t_2-t_1 \end{bmatrix} \right)$$

— III —

To extend to $[0,\infty)$:—

Take $(\Omega, \mathcal{F}, P)$ with iid $N(0,1)$, $\xi_{k,j}$, $k,j \geq 1$

Then use $\xi_{k1}, \xi_{k2} \cdots$ to Construct $(B_k(t))_{0\leq t\leq 1}$ a std BM run for time 1.

Of course $B_1(\cdot)$, $B_2(\cdot) \cdots$ are themselves independent

Then set
$$B(t) = \begin{cases} B_1(t) \ ; & 0 \leq t \leq 1 \\ B_2(t) + B_1(1), & 1 \leq t \leq 2 \\ B_{(3)}(t) + B_2(1) + B_1(1); & 2 \leq t \leq 3 \\ \cdots \end{cases}$$

Check $(B(t))_{0 \leq t < \infty}$ is a std BM

Remark :—

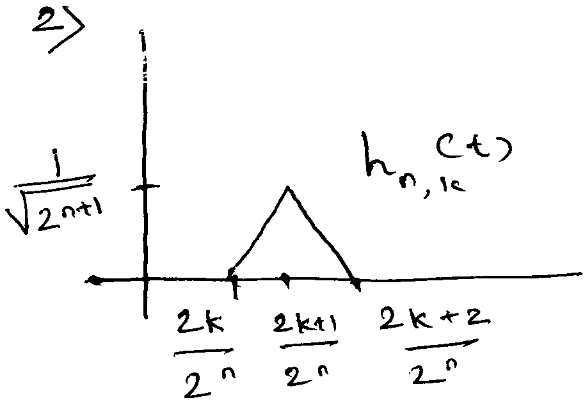
1) d-dimensional BM. :— Let $B_1, B_2, \cdots B_d$ be independent standard BMs. Then $B = (B_1, \cdots, B_d)$ is called d-dimensional BM.

(i) $B(0) = 0$ a.s.

(ii) $B(t_0),\ B(t_2) - B(t_1) \ldots$ are independent

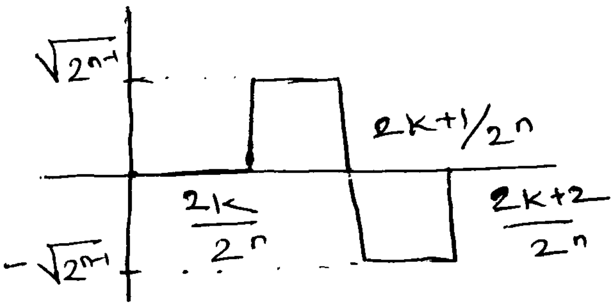
(iii) $B(s) - B(t) \sim N_d(0, (s-t)I_d)$
$t < s$.

(iv) $t \mapsto B(t)$ is a.s. cts
(from $[0,\infty) \longrightarrow \mathbb{R}^d$).

2) 

Then $F_n(t) = \sum_{k=1}^{2^{n-1}} \xi_{2^{n-1}+k} \cdot h_{n,k}(t)$

$\therefore\ B(t) = \sum_{n=0}^{\infty} \sum_{k=0}^{2^{n-1}} \xi_{2^{n-1}+k} \cdot h_{n,k}(t)$



$b_{n,k}(t)$

$\int b_{n,k}^2 = 1$

Then $\{ b_{n,k} \mid n \geq 0,\ 1 \leq k \leq 2^n \}$ is an orthonormal basis (Haar Basis). for $L^2[0,1]$

Now $\boxed{h_{n,k} = \int_0^t b_{n,k}}$

Thus
$$B(t) = \sum_{n=0}^{\infty} \sum_{k=1}^{2^n} \left(\int_0^t b_{n,k}(u)\,du \right) X_{n,k}$$

where $X_{n,k} \overset{iid}{\sim} N(0,1)$.

Note that $\int_0^t \underbrace{\sum_{n,k} X_{n,k} f_{n,k}(u)}_{\downarrow \text{ white noise}} du$

diverges a.s.

Remark (8) $D[0,\infty)$: cadlag.

Exer 1) What are the cpt subsets of $C[0,1]$?

13-8-2009

Prob $p \to$ bring a new person

$1-p \to$ drop out

$p > \frac{1}{2}$: explode

$p \leq \frac{1}{2}$: vanish

4) Continuity properties of BM – Negative results

$(B_t)_{0 \leq t \leq 1} \to$ BM (1-dim) run for unit time

Ex:* Fix $t_0 \in [0,1]$. Then $P\{B$ is differentiable at $t_0\}$ is zero.

Note $P\{B(t_0) = 1\} = 0$

$\not\Rightarrow$ $P\{B(t) = 1$, for some $t \in [0,1]\} = 0$

$\because$ this is greater than or equal to $P\{B(1) > 1\}$

Remarks:–

1) For any $D \subseteq [0,1]$ which is countable (say $D = \mathbb{Q}$)

$P\{B$ is differentiable at some $t \in D\}$

$= P\{\bigcup_{t \in D} \{B \text{ diff at } t\}\} = 0$

2) For $\omega \in \Omega$, let $E_\omega = \{t \in [0,1] / B_\omega'(t) \text{ exists}\}$

Then $Leb(E_\omega) = 0$ for a.e (ω).

$\xrightarrow[\text{Fix } t]{}$ Let $f(\omega,t) = 1_{\{B'_\omega(t) \text{ exists}\}}$

$$\int_\Omega f(\omega,t)\, dP(\omega) = P\{B'(t) \text{ exists}\}$$
$$= 0 \quad (\text{by Ex}^n)$$

$$\Rightarrow \int_0^1 \int_\Omega f(\omega,t)\cdot dP(\omega)\, dt = 0$$

Now f is non-ve $\therefore$ by applying Fubini's thm

$$\int_\Omega \int_0^1 f(\omega,t)\, dt \cdot dP(\omega) = 0$$

$$\Rightarrow \int_\Omega Leb(E_\omega)\, dP(\omega) = 0$$

$$\Rightarrow Leb(E_\omega) = 0 \quad a.s.$$

Two gaps :—

$\rightarrow$ For applying Fubini's thm, f should be jointly m'ble.
i.e $(\omega,t) :\longrightarrow B_\omega(t)$ is jtly m'ble. If so $f(\omega,t)$ is also m'ble.

$\rightarrow$ ~~Are~~ such sets like $\{B$ is diff at $t\}$ $\{B$ is diff at some $t\}$ are m'ble (in $\mathcal{B}_{[0,1]}$).

Paley Wiener-Zygmund :—

i) $(\Omega,\mathcal{F},P)$, $(B_t)_{0\le t\le 1}$. Then $P\{B$ is differentiable for some $t\in[0,1]\} = 0$.

Proof :— [Dvoretsky - Erdős - Kakutani]

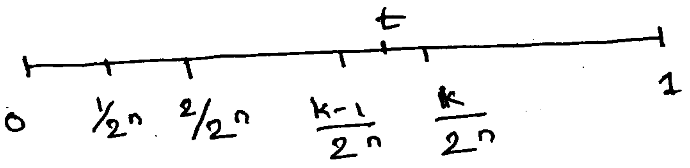
[Suppose $f : [0,1] \xrightarrow{cts} \mathbb{R}$. For some $t\in[0,1]$ $f'(t)$ exists.

$$\sup_{s\in[0,1]} \left| \frac{f(s)-f(t)}{s-t} \right| < \infty.$$
]

Enough to show, for any $M > 0$ that

$$P\underbrace{\left\{ \inf_t \sup_s \left| \frac{B_s - B_t}{s-t} \right| \le M \right\}}_{E} = 0$$

Take $n \geq 1$ and consider the dyadics



Suppose $\exists$ such that, $\frac{k-1}{2^n} \le t < \frac{k}{2^n}$

$$\left\{ \sup_s \left| \frac{B_s - B_t}{s-t} \right| \le M \right\}$$

then $\left| B\left(\frac{k}{2^n}\right) - B(t) \right| \le M\, 1/2^n$

$$\left| B\left(\frac{k+1}{2^n}\right) - B(t) \right| \le M \cdot 2/2^n$$

$$\left| B\left(\frac{k+2}{2^n}\right) - B(t) \right| \le M \cdot 3/2^n$$

$$\left| B\left(\frac{k+3}{2^n}\right) - B(t) \right| \le M\, 4/2^n$$

Then
$$\left.\left\{\begin{array}{l} \left| B\left(\frac{k+3}{2^n}\right) - B\left(\frac{k+2}{2^n}\right) \right| \le 7M/2^n \\ \left| B\left(\frac{k+2}{2^n}\right) - B\left(\frac{k+1}{2^n}\right) \right| \le 7M/2^n \\ \left| B\left(\frac{k+1}{2^n}\right) - B\left(k/2^n\right) \right| \le 7M/2^n \end{array}\right.\right\} A_{nk}$$

$$E \subseteq \bigcap_{n=1}^{\infty} \bigcup_k A_{n,k}.$$

$P\{A_{n,k}\}$

$= \left(P\left\{\left|\frac{\xi}{\sqrt{2^n}}\right| \le \frac{7M}{2^n}\right\}\right)^3$

$= \left(P\{|\xi| \le 7M2^{-n/2}\}\right)^3$

$\le \left(\frac{1}{\sqrt{2\pi}} \cdot \frac{7M}{\sqrt{2^n}}\right)^3$

$\le \frac{M'}{2^{3n/2}}$

$P\left\{\bigcup_{k=1}^{2^n-3} A_{n,k}\right\} \le \frac{2^n \cdot M'}{2^{3n/2}}$

$= \frac{M'}{2^{n/2}}$ ← Summable.

$\sum_n P\{\bigcup_{k=1} A_{n,k}\} < \infty$

Using Borel Cantelli lemma,

w.p.1 only finitely many of $\bigcup_k A_{n,k}$ happen.

$\Rightarrow P\{\bigcap_n \bigcup_k A_{n,k}\} = 0$

PWZ ii) : $(\Omega, \mathcal{F}, P)$, $(B_t)_{0 \le t \le 1}$

$P\{\forall t, \limsup_{h \to 0} \frac{|B_{t+h} - B_t|}{h^\alpha} = \infty\} = 1 \quad \forall \alpha > \frac{1}{2}$

Proof: Proof is almost same

In Event E we make a slight modificat^n

i.e instead of $(s-t)$, we have $(s-t)^\alpha$.

In $A_{n,k}$ we add $\left|B\left(\frac{k+4}{2^n}\right) - B\left(\frac{k+3}{2^n}\right)\right| \le \frac{M'}{2^{n\alpha}}$, Then we get

$$P\{A_{n,k}\} = \left(P\{|\xi| \le 7M\,2^{-n(\alpha-1/2)}\}\right)^4$$

$$\le \frac{M''}{2^{4(\alpha-1/2)n}}$$

$$\therefore\ P\left\{\bigcup_{k=1}^{2^n} A_{n,k}\right\} \le \frac{M'}{2^{[4(\alpha-1/2)-1]n}}$$

→ summable if $\alpha - \frac{1}{2} > 1/4$ i.e. $\alpha > \frac{3}{4}$

Ill'ly inc the no of intervals we can make $\alpha > 1/2$.

Ex: i) Complete the proof for $\alpha > 1/2$

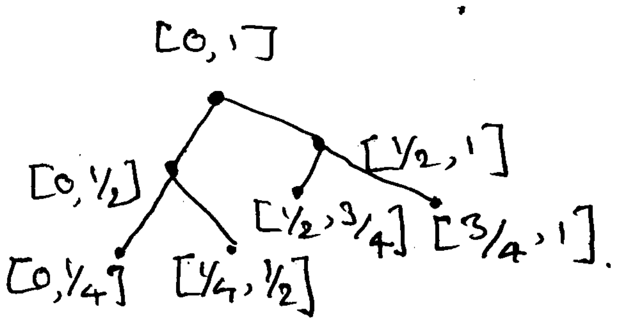
ii) what happens if $\alpha = 1/2$?

For $\alpha = 1/2$, $\exists\, c_0 > 0$

$$P\left\{\forall t,\ \limsup_{h\to 0} \frac{|B_{t+h} - B_t|}{\sqrt{h}} \ge c_0\right\} = 1$$

→ Dvoretzky.

Corollary: (Paley-Wiener-Zygmund)



Binary tree.

If I, J are dyadic intervals then $I^\circ \cap J^\circ = \phi$ or $I \subseteq J$ or $J \subseteq I$.

__Notation:__— For any interval $I = [s,t]$

Let $B(I) = B_t - B_s$

__Proof:__— Enough to show that for any fixed $\varepsilon > 0$.

$$P\{\exists t \in [0,1] \text{ s.t } |B_s - B_t| \le C_0 |s-t|^{k} \\ \forall s \in [t-\varepsilon, t+\varepsilon]\} = 0.$$

[Note if $\limsup_{s \to t} \frac{|B_s - B_t|}{\sqrt{|s-t|}} < C_0$

then $\exists\ \varepsilon > 0$ s.t

$$\frac{|B_s - B_t|}{\sqrt{|s-t|}} < C_0 \quad \forall s \in [t-\varepsilon, t+\varepsilon] \Big)$$

Let us fix C_0, ε.

For a dyadic interval I

if $|B(I)| < C_0\sqrt{|I|}$ colour them green.

Colour them blue if

$$|B(I)| \le C_0\sqrt{|I|}$$

$$\& \quad |B(J)| \le C_0\sqrt{|J|}.$$

where J is the parent of I

18-08-2009

Thm:— $P\left\{ \inf_{t\in[0,1]} \limsup_{h\to 0} \dfrac{|B_{t+h}-B_t|}{\sqrt{h}} < \dfrac{1}{10} \right\} = 0$

Proof: The event in question is the same as

$$\bigcup_{M=1}^{\infty} \left\{ \exists\, t \in [0,1] \text{ s.t. } |B_{t+h}-B_t| < \tfrac{1}{10}\sqrt{h},\ \forall\, h < 1/M \right\} := A_M$$

Enough to show that

$$P\{A_M\} = 0,\ \forall\, M = 1, 2, \dots$$

Fix M

Let $S_0 = \{[0,1]\}$

For $k \geq 1$.

Let $S_k = \left\{ I_{j,k} = \left[\frac{j-1}{2^k}, \frac{j}{2^k}\right],\ 1 \leq j \leq 2^k \,/\, |B(I_{j,k})| \leq \frac{1}{10}\sqrt{2^{-k}},\ \text{parent}(I_{j,k}) \in S_{k-1} \right\}$.

Clearly $S_0, S_1 \dots$ is a 'branching process'.

We want to show that it dies out (w.p.1)

Condition on $S_0, S_1 \dots S_{k-1}$

Note that $S_0, S_1 \dots S_{k-1}$ are functions of $\{B(j/2^{k-1}) \mid 0 \leq j \leq 2^{k-1}\}$

I

$J \quad K$

Condition on $\{B(j/2^{k-1}) \,/\, 0 \leq j \leq 2^{k-1}\}$ $\left[\frac{2j}{2^k}, \frac{2j+1}{2^k}\right]$ $\frac{2j+1}{2^k}, \frac{2j+2}{2^k}$

Now consider any $I = \left[\frac{j}{2^{k-1}}, \frac{j+1}{2^{k-1}}\right]$

we want to find $P\{J \in S_k \mid B(j/2^{k-1}), 0 \leq j \leq 2^{k-1}\}$

Given $B(j/2^{k-1})$, $0 \le j \le 2^{k-1}$

We can write

$$B\left(\frac{2j+1}{2^k}\right) = \left[\tfrac{1}{2} B\left(\frac{j}{2^{k-1}}\right) + \frac{1}{2} B\left(\frac{j-1}{2^{k-1}}\right) + \frac{\xi_{k,j}}{\sqrt{2^{k+1}}}\right.$$

where ξ_{kj} are iid $N(0,1)$

Hence $B(J) = \frac{1}{2} B(I) + \frac{\xi_{k,j}}{\sqrt{2^{k+1}}}$

and $|B(K)| = \frac{1}{2} B(I) - \frac{\xi_{k,j}}{\sqrt{2^{k+1}}}$

Now consider

$$P\left\{ |B(J)| \le \frac{\sqrt{2^{-k}}}{10} \Big/ B_{j/2^{k-1}},\ 0 \le j \le 2^{k-1} \right\}$$

$$= P\left\{ \left| \tfrac{1}{2} B(I) + \frac{\xi_{kj}}{\sqrt{2^{k+1}}} \right| \le \frac{\sqrt{2^{k}}}{10} \Big/ \text{—}\text{''}\text{—} \right\}$$

$$= P\left\{ \left| \frac{\sqrt{2^{k+1}}}{2} B(I) + \xi_{kj} \right| \le \frac{\sqrt{2}}{10} \Big/ \text{—}\text{''}\text{—} \right\}$$

$$\le \frac{2\sqrt{2}}{\sqrt{2\pi}\cdot 10} = \frac{1}{5\sqrt{\pi}}$$

Hence i) E(# of offsprings of I to survive in S_k $\mid S_0, \dots S_{k-1}\}$.

$$\le \frac{2}{5\sqrt{\pi}}$$

(ii) # of offspring of $I_{j,k-1}$ are independent as j varies conditional on $S_0, \dots S_{k-1}$

Hence $E[\# S_k / S_0 \dots S_{k-1}] \leq \frac{2}{5\sqrt{\pi}} (\# S_{k-1})$

$\Rightarrow E[\# S_k] \leq \frac{2}{5\sqrt{\pi}} \cdot E[\# S_{k-1}]$

$\leq \dots \leq \left(\frac{2}{5\sqrt{\pi}}\right)^k \longrightarrow 0 \quad k \to \infty$

Thus $P\{S_k \neq \phi\} \leq E[\# S_k] \to 0$

X is r.v. with values in $0, 1, 2, \dots$

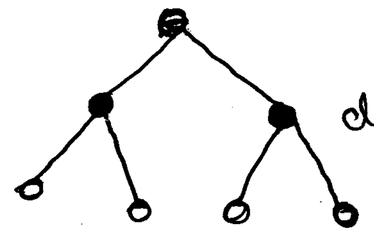
$P\{X \geq 1\} \leq E[X]$

$\sum_{j=1}^{\infty} P\{X = j\} \leq \sum_{j=1}^{\infty} j\, P\{X = j\}$.

$\therefore$ P{ Branching process survives forever} $= 0$

$\| \bigcap_k \{S_k \neq \phi\} \|$.

$A_M = \{\exists t \text{ s.t } |B_{t+h} - B_t| \leq \frac{1}{10}\sqrt{|h|} \ \forall\ 0 \leq |h| \leq \frac{1}{M}\}$



dead

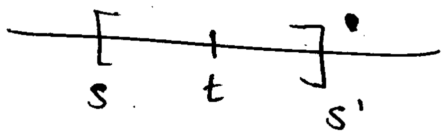
clearly one need not have to start from S_0. One can start from some where in later stage.

Let K_0 be large enough so that $2^{-K_0} < \frac{1}{2M}$

Let S_{K_0} = all dyadic intervals of length 2^{-K_0}

Run the branching process from K_0 onwards

$$[s, s'] \subseteq \left[t - \tfrac{1}{M}, t + \tfrac{1}{M}\right]$$



$$|B_s - B_{s'}| \le |B_s - B_t| + |B_{s'} - B_t|$$
$$\le \frac{1}{10}\sqrt{(s-t)} + \frac{1}{10}\sqrt{|s'-t)}$$
$$\le \frac{2}{10}\sqrt{|s-s'|}$$

Blue if $|B(I)| \le \frac{2}{10}\sqrt{|I|}$ and parent in S_{K-1}

Hence $P\{A_M\} = 0$

<u>Exer:</u> Find the best c_0 (in this proof) such that

$$P\left\{ \inf_t \limsup_{h \to 0} \frac{|B_{t+h} - B_t|}{\sqrt{h}} < c_0 \right\} = 0 \longrightarrow (*)$$

<u>Remarks:</u> i) Optimal c_0 is 1 i.e. actually the statement $(*)$ holds for $c_0 = 1$ but not for $c_0 > 1$.

5) Continuity properties: Positive results

Paley Wiener-Zygmund :— $(B_t)_{t \le 1}$ — BM.

Then for any $\alpha < \frac{1}{2}$,

$$P\left\{ \sup_{s \neq t} \frac{|B_t - B_s|}{|t-s|^\alpha} < \infty \right\} = 1$$

Proof :— Fix n and consider dyadic points

$\frac{k}{2^n}$, $0 \le k \le 2^n$.

$$B\left(\frac{k+1}{2^n}\right) - B\left(\frac{k}{2^n}\right) \sim N(0,1)$$

$$P\left\{ \left| B\left(\frac{k+1}{2^n}\right) - B\left(\frac{k}{2^n}\right) \right| > \frac{1}{2^{n\alpha}} \right\}$$

$$= P\left\{ \frac{|X|}{\sqrt{2^n}} > \frac{1}{2^{n\alpha}} \right\}$$

$$= P\left\{ |X| > 2^{n(\frac{1}{2}-\alpha)} \right\}$$

Hence $P\left\{ \max_{1 \le \le 2^n} \left| B\left(\frac{k+1}{2^n}\right) - B\left(\frac{k}{2^n}\right) \right| > \frac{1}{2^{n\alpha}} \right\}$

$$\le 2^n \, P\left\{ |X| > 2^{n(\frac{1}{2}-\alpha)} \right\}$$

$$\le 2^n e^{-2^{n(1-2\alpha)-1}} \quad \text{(summable)}$$

Borel Cantelli $\Rightarrow$ $\exists N = N(\omega) < \infty$ a.s. $\max_{1 \le k \le 2^n} \left| B\left(\frac{k+1}{2^n}\right) \right| < \frac{1}{2^{n\alpha}}$,

$\forall n \ge N$

Now fix any $t < s$

Let m be such that $2^{-m} < s - t < 2^{-m+1}$

Then $\exists$ at least one dyadic pt $\frac{k}{2^m}$ between

$$t \le \underset{\underset{u}{\shortparallel}}{\frac{k}{2^m}} \le s$$

$$|B_s - B_t| \le |B_s - B_u| + |B_u - B_t|$$

$$\begin{array}{ccccc} t & u & u_1 & u_2 & s \\ & \shortparallel & \shortparallel & \shortparallel & \\ & \frac{k}{2^m} & \frac{k_1}{2^{m_1}} & \frac{k^2}{2^{m_2}} & \end{array}$$

$$|B_s - B_u| \le |B_{u_1} - B_u| + |B_{u_2} - B_{u_1}| + \cdots$$

$$\le \frac{1}{2^{m\alpha}} + \frac{1}{2^{(m+1)\alpha}} + \frac{1}{2^{(m+2)\alpha}}$$

$$\le \frac{c}{2^{m\alpha}}.$$

20-8-2009

Let $c(\omega) = \max\limits_{1 \le n < N(\omega)} \max\limits_{1 \le k \le 2^n} 2^{n\alpha} \left| B\left(\frac{k+1}{2^n}\right) - B\left(\frac{k}{2^n}\right) \right|$

Then $c < \infty$ w.p. 1 and

$$\left| B_\omega\left(\frac{k+1}{2^n}\right) - B_\omega\left(\frac{k}{2^n}\right) \right| \le \frac{c(\omega)}{2^{n\alpha}} \quad \forall n \ge 1 \quad \forall\ 1 \le k \le 2^n$$

<u>Observation:</u> — Let $0 \le t < s < 1$

Consider the dyadics $u \in [t, s]$ with the smallest denominator. There is unique such u.
For, suppose $u_1 = \frac{k_1}{2^m}$, $u_2 = \frac{k_2}{2^m}$ are 2 such with $k_1 < k_2$, then some $1 \le 2j \le k_2$ and
$\frac{2j}{2^m} = \frac{j}{2^{m-1}} \in [t, s]$. —

Now take any $0 \le t < s \le 1$
$u_0 :=$ the "smallest dyadic" in $[t, s] = \frac{k_0}{2^{m_0}}$
$u_1 :=$ the " —— " —— " in $[u_0, s] = \frac{k_1}{2^{m_1}}$
$u_2 :=$ —— " —— in $[u_1, s] = \frac{k_2}{2^{m_2}}$

If s is dyadic, then $u_k = s$ for some k and we stop there)

Since dyadics are dense, $u_k \to s$

Hence
$$\left| B(s) - B(u_0) \right| = \left| \sum_{k=1}^{\infty} [B(u_k) - B(u_{k-1})] \right| \quad \text{—} (*)$$

Now note that

$$\begin{array}{ccc} \hline u_j & u_{j+1} & s \end{array}$$

$u_j = \frac{k_j}{2^{m_j}}$ $\quad \frac{k_{j+1}}{2^{m_{j+1}}}$

$m_{j+1} > m_j$

$u_j = \frac{2^{m_{j+1} - m_j} k_j}{2^{m_{j+1}}} < u_{j+1} = \frac{k_{j+1}}{2^{m_{j+1}}}$

$\Rightarrow k_{j+1} = 2^{m_{j+1} - m_j} k_j + 1$

Thus from the note we can conclude that

1) $m_{j+1} > m_j$

2) $u_j = \dfrac{k_j}{2^{m_j}} = \dfrac{l_j}{2^{m_{j+1}}}$ for some l_j

$u_{j+1} = \dfrac{k_{j+1}}{2^{m_{j+1}}}$ then $k_{j+1} = l_{j+1} + 1$.

3) Hence $|B_\omega(u_{j+1}) - B_\omega(u_j)|$

$$\leq \frac{c(\omega)}{2^{\alpha m_{j+1}}}.$$

$\therefore$ from $(*)$ we have

$$|B(s) - B(u_0)| = \left| \sum_{k=1}^{\infty} [B(u_k) - B(u_{k-1})] \right|$$

$$\leq \sum_{k=1}^{\infty} \frac{c(\omega)}{2^{\alpha m_k}}$$

$$\leq \frac{c(\omega)}{} \left[\frac{1}{2^{m_0 \alpha}} + \frac{1}{2^{(m_0+1)\alpha}} + \cdots \right]$$

$$\leq \frac{c'(\omega)}{2^{m_0 \alpha}} \quad \text{where} \quad c'(\omega) = \frac{c(\omega)}{1 - \frac{1}{2^\alpha}}.$$

Similar consideration give

$$|B_\omega(u_0) - B_\omega(t)| \leq \frac{c(\omega)}{2^{m_0 \alpha}}.$$

Therefore
$$|B_\omega(s) - B(t)| \le \frac{2C'(\omega)}{2^{m_0 \alpha}}$$

~~By choice of m_0, $2^{-m_0} \le s-t$.~~

Choice of r_0; r_1 ... needs a revision.

Pick m_0 s.t
$$2^{-m_0} \le |s-t| < 2^{-m_0+1}$$

There will be $r_0 = \frac{k_0}{m_0} \in [t,s]$, there are at most 2 choices for r_0.

Take the right one.
$$r_0 = \frac{k_0}{2^{m_0}} = \frac{2^l k_0}{2^{m_0+l}}$$

Find the smallest l s.t
$$r_1 = \frac{2^l k_0 + 1}{2^{m_0+l}} \le s$$
$$\parallel$$
$$\frac{k_1}{2^{m_1}}$$

Then find the smallest l s.t
$$r_2 = \frac{2^l k_1 + 1}{2^{m_1+l}} \le s$$
$$\parallel$$
$$\frac{k_2}{2^{m_2}}$$

and so on. All the arguments will follow.

Therefore

$$|B_\omega(s) - B_\omega(t)| \le \frac{2C'(\omega)}{2^{m_0\alpha}} \le C''(\omega)|t-s|^\alpha$$

$\blacksquare$

Note one cannot find uniform Hölder constant in case $[0,\infty)$.

Remarks:— 1) For a cts $f:[0,1]\to\mathbb{R}$ modulus of continuity

$$\omega_f(\delta) = \sup\{|f(s)-f(t)| : |s-t|\le\delta\}$$

($\omega_f(\delta)\downarrow 0$ as $\delta\downarrow 0$).

PWZ-II shows that

$$\frac{\omega_B(\delta)}{\delta^\alpha} \longrightarrow 0 \quad \text{w.p.1 for any } \alpha<\tfrac{1}{2}$$

(why?)

Result of Levy:—

$$\limsup_{\delta\downarrow 0} \frac{\omega_B(\delta)}{\sqrt{2\delta\log(\frac{1}{\delta})}} \longrightarrow 1 \quad \text{w.p.1}$$

Ex:* (i) Is Wiener measure compactly supported?

(ii) Show that given $\varepsilon>0$, $\exists\ K \overset{cpt}{\subseteq} C[0,1]$ such that Wiener measure of $K \geq 1-\varepsilon$

25-8-2009

6: Two measure questions

1) Completion: $(\Omega, \mathcal{F}, P) \to$ a prob space

Let $\mathcal{N} = \{A \subseteq \Omega \,/\, \exists B \in \mathcal{F},\ B \supseteq A$ and $P(B) = 0\}$
(null sets)

Let $\bar{\mathcal{F}} = \sigma\{\mathcal{F} \cup \mathcal{N}\}$

~~Set P~~

Ex: There is a unique prob. measure $\bar{P}$ on $(\Omega, \bar{\mathcal{F}})$
such that $\bar{P}(A) = \begin{cases} P(A), & \text{if } A \in \mathcal{F} \\ 0, & \text{if } A \in \mathcal{N} \end{cases}$

Hint: $\cdot\ \bar{\mathcal{F}} = \{A \cup B \,/\, A \in \mathcal{F},\ B \in \mathcal{N}\}$

Now onwards σ field is complete.

2) Joint measurability of $B(t, \omega)$:- $(B_\omega(t))_{0 \le t \le 1},\ \omega \in \Omega$

Recall the construction of BM on $(\Omega, \mathcal{F}, P)$ using approximations B_n

$B_\omega =$ uniform limit of $B_{n,\omega}$

We now consider $(t, \omega) \longmapsto B_\omega(t)$

$$[0,1] \times \Omega \longmapsto \mathbb{R}$$

Let $x \in \mathbb{R}$

$$\{(t, \omega) \,/\, B_\omega(t) < x\}$$

$$= \bigcup_{N=1}^{\infty} \bigcap_{n \ge N} \{(t, \omega) \,/\, B_{n,\omega}(t) < x\}$$

Ex*: Check that for any n, $x \in \mathbb{R}$, the set $\{(t, \omega) \,/\, B_{n,\omega}(t) < x\}$ is in the product σ-field $\mathcal{B} \otimes \mathcal{F}$

$\downarrow$

Borel σ-field on $[0,1]$

7 : Invariance properties of BM

$(C[0,\infty), \mathcal{B}, \mu_{wie})$
$\downarrow$
Wiener measure

A measurable transformation $T : C[0,\infty) \to C[0,\infty)$ is said to be μ preserving (or we say μ_{wie} is invariant under T) if $\mu T^{-1} = \mu$. —— (1)

Equivalently $(\Omega, \mathcal{F}, P)$, $(B_t)_{t \geq 0} = B$ std BM.

$$\begin{tikzcd} \Omega \arrow[r, "B"] \arrow[r, bend right, "W = T\circ B"'] & C[0,\infty) \arrow[loop right, "T"] \end{tikzcd}$$

Let $W = T \circ B$.

(1) is same as saying that W and B have the same distribution ($W \overset{d}{=} B$).

e.g.

(1) $Tf = -f$, i.e. $W_t = -B_t$. Then $W \overset{d}{=} B$

Proof :— a) $\omega_0 = -B_0 = 0$ w.p 1.

(b) $(\omega_{t_1}, \omega_{t_2} - \omega_{t_1}, \ldots, \omega_{t_K} - \omega_{t_{K-1}})$

$= (-B_{t_1}, -B_{t_2} + B_{t_1}, \ldots \ldots -B_{t_K} + B_{t_{K-1}})$

$0 \leq t_1 < t_2 \cdots < t_K$

$\because B_{t_1}, B_{t_2} - B_{t_1}$ are indep, so are $W_{t_1}, W_{t_2} - W_{t_1} \cdots$

c) $t < s$, $\omega_s - \omega_t = -(B_s - B_t) \sim N(0, s-t)$

d) $t \to \omega_t = -B_t$ is cts for a.e (ω)

—■—

2) Scaling invariance :— Let $a > 0$. Let $W_t = aB\left(\frac{t}{(a)^2}\right)$ then $W \overset{d}{=} B$

$T : C[0,\infty) \longrightarrow C[0,\infty)$

$\longleftrightarrow$ $(Tf)(t) = af(t/a^2)$

3) Time reversal :—

Let $W_t = \begin{cases} tB(1/t) & t>0 \\ 0 & t=0 \end{cases}$

$\longleftrightarrow \quad Tf(t) = \begin{cases} tf(1/t), & t>0 \\ 0 & \text{if } t=0. \end{cases}$

4) Shift invariances — Fix $T>0$

Define $Tf(t) = f(T+t) - f(T)$, $t \geq 0$

or $W_t = B(t+T) - B(T)$, $t \geq 0$.

Then $W \overset{d}{=} B$.

~~Obs~~

Defn : Let $X = (X_t)_{t\geq 0}$ $(\Omega, \mathcal{F}, P)$ be a $C[0,\infty)$ valued r.v. We say that X is a Gaussian process if for any $0 \leq t_1 < t_2 < \cdots < t_n$ the vector $(X_{t_1}, X_{t_2}, \ldots X_{t_n})$ has a normal distribution.

Let $m(t) = E[X_t]$ — mean function

$K(t,s) = \cancel{E[X_t X_s]}$ — covariance kernel
$E[(X_t - m(t))(X_s - m(s))]$

Remarks — If X and Y are two Gaussian processes with the same $m(\cdot)$ and $K(\cdot,\cdot)$ then $X \overset{d}{=} Y$

Reason By assumption, X and Y have same finite dimensional distributions and f.d. cylinders generate $\mathcal{B}_{[0,\infty)}$

Eg: $(B_t)_{t\geq 0}$ is a Gaussian process

$m(t) = E[B(t)] = 0$

$K(t,s) = E[B(t)B(s)] = t \wedge s$ $\boxed{\wedge := \min}$

Proof of invariance property (IP) (2):

W is clearly a cts Gaussian process.

$E(W_t] = E[aB(t/a^2)] = 0$

$E(W_t W_s] = a^2 E B(t/a^2) B(s/a^2)]$

$= a^2 \left[\frac{t}{a^2} \wedge \frac{s}{a^2}\right]$

$= t \wedge s$

$\Rightarrow W \stackrel{d}{=} B$

Proof of IP: 3. $E[W_t] = \begin{cases} tE[B(1/t)] = 0, & \text{if } t>0 \\ 0, & \text{if } t=0. \end{cases}$

cov: $E[W_t W_s] = ts E[B(1/t), B(1/s)]$, if $t>0, s>0$

$= ts(1/t \wedge \frac{1}{s})$

$= t\wedge s$

$E[W_t W_0] = 0 \quad \forall t$

$= 0 \wedge t$

Need only to prove that $t \mapsto W_t$ is a.s. cts.

clearly $(W_t)_{t>0}$ is a.s cts

i.e to TPT $W_t \xrightarrow{a.s} 0$ as $t\to 0$

$\omega_t \to 0 \iff \frac{B_s}{s} \to 0$

as $t \to 0$ as $s \to \infty$

$\frac{B_s}{s} \sim N(0, \frac{1}{s})$

$\Rightarrow \frac{B_s}{s} \xrightarrow{d} 0$ as $s \to \infty$

$\Rightarrow \frac{B_s}{s} \xrightarrow{P} 0$ as $s \to \infty$

or $\sum P\{|\frac{B_n}{n}| > \epsilon\} = \sum_n P\{|X| > \epsilon\sqrt{n}\}$

$\leq \sum_n e^{-\epsilon^2 n/2}$

$< \infty$

$\frac{B_n}{n} \xrightarrow{a.s.} 0 \quad n \to \infty$ by B.C. lemma.

Consider $(0, \infty)$

$(W_t)_{t>0}$ and $(B_t)_{t\geq 0}$ are then equal in distribution.

$(\omega_t)_{t>0} \stackrel{d}{=} (B_t)_{t>0}$

Then $\{W_t \to 0 \text{ as } t \to 0\} = \bigcap_{\substack{\epsilon>0 \\ \epsilon \in \mathbb{Q}}} \bigcup_{\substack{\delta>0 \\ \delta \in \mathbb{Q}}} \bigcap_{\substack{0<t<\delta \\ t \in \mathbb{Q}}} \{|W_t| < \epsilon\}$

$\{B_t \to 0 \text{ as } t \to 0\} = \bigcap \bigcup \bigcap \{|B_t| < \epsilon\}$

R.H.S are defined in terms of $(W_t)_{t>0}$ and $(B_t)_{t>0}$

Hence must have the same probability.

$\therefore P\{B_t \to 0 \text{ as } t \to 0\} = 1$ & hence

$P\{\omega_t \to 0 \text{ as } t \to 0\} = 1$. $\because$ B is a.s. cts at 0.

Ex*: ~~Prove shift invariance~~

Fix $T > 0$ and let $\alpha < \frac{1}{2}$.

$$C_T = \sup \left\{ \frac{|B_t - B_s|}{|t-s|^\alpha} \;\middle|\; t \neq s,\ t \leq T,\ s \leq T \right\}$$

What is the relationship between the distributions of C_T and C_1.

Ex: Check that all these hold for d-dim BM.

Planar B.M: Suppose it is scaled differently in diff. places.

[Now $f : \mathbb{C} \to \mathbb{C}$ is entire then

$f(z) = \sum_{n=0}^{\infty} c_n z^n$, $f(z_0+h) \approx f(z_0) + hf'(z_0) + O(h^2)$]

Now $(B_t)_{t \geq 0}$, $W_t = f(B_t)$

Then we shall see that $(W_{\theta(t)})_{t \geq 0} \stackrel{d}{=} (B_t)_{t \geq 0}$.

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Proof for shift invariance:—

$t < s$, $E[W_t W_s] = E[(B_{T+t} - B_T)(B_{T+s} - B_T)]$

$= (T+t) \wedge (T+s) - (T+t) \wedge T - T \wedge (T+s) + T \wedge T$

$= T+t - T - T + T$

$= t$

$\Rightarrow W \stackrel{d}{=} B$

Ex :— Rotational invariance of BM :— Let $d \geq 2$ and $B = (B_1, \dots B_d)$ be a std d-dim B.M. Fix $A_{d \times d}$ real matrix. Let $W = AB$ i.e

$$\begin{bmatrix} W_1(t) \\ \vdots \\ W_d(t) \end{bmatrix} = A_{d\times d} \begin{bmatrix} B_1(t) \\ \vdots \\ B_d(t) \end{bmatrix}$$

Show that $W \overset{d}{=} B$ iff $A^T A = I$

Note that

$$W_t = B(T+t) - B(T), \quad t \geq 0$$

$$X_t = B_t \quad , \quad 0 \leq t \leq T$$

$$s < T$$

Then $X_s = B_s$ & $W_{t_1} = B_{t_1+T} - B_T$ are independent. This motivate us for looking in Markov property.

8. σ-fields Filtrations etc :—

$(\Omega, \mathcal{F}, P)$ - a prob space. Suppose $\mathcal{G}_1, \mathcal{G}_2$ are two sub σ-fields of $\mathcal{F}$. We say $\mathcal{G}_1$ and $\mathcal{G}_2$ are independent if $P(A \cap B) = P(A) \cdot P(B)$ $\forall A \in \mathcal{G}_1$ $\forall B \in \mathcal{G}_2$.

Eg: $\mathcal{G}_1 = \sigma(X_i : i \in I)$

$\mathcal{G}_2 = \sigma(Y_j : j \in J)$

Then $\mathcal{G}_1$ and $\mathcal{G}_2$ are independent $\iff$

$(X_i : i \in I)$ and $(Y_j : j \in J)$ are independent $\Leftrightarrow$ $(X_{i_1}, \cdots X_{i_n})$ is independent of $(Y_{j_1}, \cdots Y_{j_n})$ for any $i_1, \cdots i_n \in I$ and $j_1 \cdots j_n \in J$.

Ex: If Y is measurable w.r.t $\sigma(X)$ then Y is a f^n of X.

Markov property of BM:—

$(\Omega, \mathcal{F}, P)$, $(B_t)_{t \geq 0}$ std BM. Fix T

Let $W_t = B_{T+t} - B_t$, $t \geq 0$.

Let $\mathcal{F}_T^0 = \sigma(B_s \mid s \leq T)$

(i) W is std BM

(ii) W is independent of $\mathcal{F}_T^0$

1) Filtration:— $(\Omega, \mathcal{F}, P)$. Let $\mathcal{F}_t$, $t \geq 0$ be sub σ-fields of $\mathcal{F}$ such that if $s < t$ then $\mathcal{F}_s \subseteq \mathcal{F}_t$. Then we say that $\mathcal{F}_\bullet = (\mathcal{F}_t)_{t \geq 0}$ is a filtration.

We say that $(\Omega, \mathcal{F}, \mathcal{F}_\bullet, P)$ is a filtered prob. space

2) Let $(\Omega, \mathcal{F}, \mathcal{F}_\bullet, P)$ be F.P.S. Let $X = (X_t)_{t \geq 0}$ be a stochastic process on $(\Omega, \mathcal{F}, P)$ (i.e X_t is r.v. in $(\Omega, \mathcal{F}, P)$). Then we say that X is adapted to $\mathcal{F}_\bullet$ if $X_t \in \mathcal{F}_t$ $\forall t$. Equivalently $\sigma(X_s \mid s \leq t) \subseteq \mathcal{F}_t$

Eg:- If $\mathcal{G}_t = \sigma(X_s / s \leq t)$ then X is adapted to $\mathcal{G}$. Infact if $X_\cdot$ is also adapted to $\mathcal{F}$. then $\mathcal{F}_t \supseteq \mathcal{G}_t$

3) Stopping times:- $(\Omega, \mathcal{F}, \mathcal{F}_\cdot, P)$ - F.P.S.

A r.v. in $(\Omega, \mathcal{F}, P)$

$\tau : \Omega \longrightarrow [0, \infty]$ is called an $\mathcal{F}_\cdot$

stopping time if $\{\tau \leq t\} \in \mathcal{F}_t \;\; \forall t < \infty$

$\{\tau \leq t\} = \{\omega / \tau(\omega) \leq t\}$.

$\Longleftrightarrow X_t = 1_{(\tau \leq t)}$, then $X_\cdot$ is adapted to $\mathcal{F}_\cdot$

Enlarging filtrations:- $(\Omega, \mathcal{F}, \mathcal{F}_\cdot, P)$ F.P.S.

1) Define $\mathcal{F}_t^+ = \bigcap_{s>t} \mathcal{F}_s$ Then $\mathcal{F}_t^+ \supseteq \mathcal{F}_t \;\; \forall t$

$(\because \forall s > t, \mathcal{F}_s \supseteq \mathcal{F}_t)$

$\mathcal{F}^+$ is right cts in the sense that

$\mathcal{F}_t^+ = \bigcap_{s>t} \mathcal{F}_s^+ \;\; \forall t$ (i.e $(\mathcal{F}^+_\cdot)^+ = \mathcal{F}_\cdot^+$)

Eg: 1 $(C[0,\infty), \mathcal{B}, P_{wie})$ Let $\mathcal{F}_t^0 = \sigma\{\omega(s) / s \leq t\}$ $(= \sigma(X_s / s \leq t))$ → $\mathcal{F}_\cdot^0$ is a filtration

$(X_t(\omega) = \omega(t))$.

$\mathcal{F}_t^+ = \bigcap_{s>t} \mathcal{F}_s^0$. Is $\mathcal{F}_t^+ = \mathcal{F}_t^0$?

Eg: 2 $(\Omega, \mathcal{F}, P)$ $(B_t)_{t \geq 0}$ std. BM

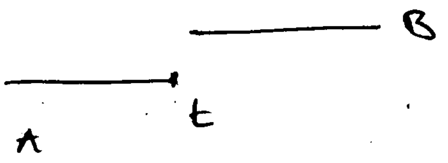
$\mathcal{F}_t^0 = \sigma(B_s / s \leq t)$ $\mathcal{F}_t^+ = \bigcap_{s>t} \mathcal{F}_s^0$

Q: $\mathcal{F}_t^+ = \mathcal{F}_t^0$?

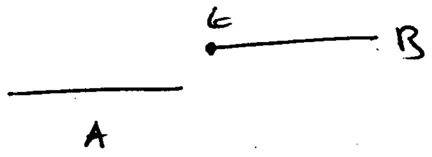
$A = \{\omega / B'_\omega(t) \text{ exists}\}$

Eg. $A \subseteq B$

$\mathcal{F}^0$:



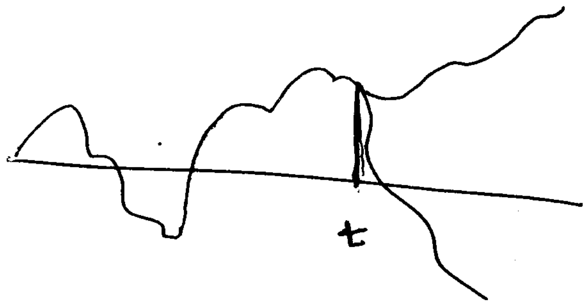
$\mathcal{F}^+_{\cdot}$:



$\therefore \mathcal{F}^0 \neq \mathcal{F}^+$

Note: If $A = \{\omega \in C[0,\infty) / \omega'(t_0) \text{ exists}\}$

$A \in \mathcal{F}^0_{t_0}$ but $A \in \mathcal{F}^0_s \ \forall s > t_0 \Rightarrow A \in \mathcal{F}^+_t$



cty of process & doesn't ensure rt cty of the filtration.

2) Completion: $(\Omega, \mathcal{F}, P)$ $\mathcal{F}.$ FPS

WLOG assume $(\Omega, \mathcal{F}, P)$ is complete

Let $N = \{A \in \mathcal{F} / P(A) = 0\}$

Define $\bar{\mathcal{F}}_t = \sigma\{\mathcal{F}_t \cup N\}$. $\bar{\mathcal{F}}_{\cdot}$ is a filtration & $\bar{\mathcal{F}}_t \supseteq \mathcal{F}_t$

Thm:- $(\Omega, \mathcal{F}, \mathcal{F}_{\cdot}, P)$ F.P.S. Then $\overline{\mathcal{F}^+_t} = \bar{\mathcal{F}}^+_t$

Proof: $\mathcal{F}_t \subseteq \mathcal{F}_s \quad s > t$

$\mathcal{F}_t^+ \subseteq \mathcal{F}_s \quad \forall s > t$

$\overline{\mathcal{F}_t^+} \subseteq \overline{\mathcal{F}_s} \quad \forall s > t$

$\overline{\mathcal{F}_t^+} \subseteq \bigcap_{s>t} \overline{\mathcal{F}_s} = \overline{\mathcal{F}}_t^+$

Suppose $B \in \overline{\mathcal{F}}_t^+$

$\iff B \in \overline{\mathcal{F}_s} \quad \forall s > t$

$\iff B = A_s \cup N_s$ where $A_s \in \mathcal{F}_s$, $N_s \in \mathcal{N}$

Take $s = t + \frac{1}{k}$ and write A_k, N_k for A_{s_k}, N_{s_k}

$\bigcup_k N_k \in \mathcal{N}$

$A = \bigcap_k A_k \in \mathcal{F}_t^+$

$B \setminus A \subseteq \bigcup_k N_k$

$P(\bigcup_k N_k) = 0$ & $\mathcal{F}$ is complete

$\Rightarrow B \setminus A \in \mathcal{N}$

Then $B = A \cup (B \setminus A)$

$\Rightarrow B \in \overline{\mathcal{F}_t^+}$

— ■ —

$\Leftarrow$: Let $B \in \overline{\mathcal{F}}_t^+$ 1-9-2009

$\Rightarrow B \in \overline{\mathcal{F}}_s, \forall s > t$

$\Rightarrow B = A_K \cup N_K, A_K \in \mathcal{F}_{t+1/K}; N_K \in N$

Let $A = \liminf_{K \to \infty} A_K = \bigcup_{N=1}^{\infty} \bigcap_{K=N}^{\infty} A_K$

$= \{\omega / \omega \in$ all but finitely many $A_K\}$

$\Rightarrow A \in \mathcal{F}_t^+$

$B = \underset{\mathcal{F}_t^+}{\underset{\cap}{A}} \cup \underset{N}{\underset{\cap}{(B/A)}}$ ($\because B/A \subseteq \bigcup_K N_K$, $P(B/A) = 0$)

$\Rightarrow B \in \overline{\mathcal{F}_t^+}$

(Here we assumed $(\Omega, \mathcal{F}, P)$ is complete.)

Note

Fix t_0

$\{\omega / \omega'(t_0)$ exists$\} \in \mathcal{F}_{t_0}^+$

$\notin \mathcal{F}_{t_0}$

$\{\omega / \lim_{n \to \infty} \frac{\omega(t_0 + 1/n) - \omega(t_0)}{1/n}$ exists$\}$

$\in \mathcal{F}_{t_0}^+$

$\notin \mathcal{F}_{t_0}^{c}$

8: Markov property: $(\Omega, \mathcal{F}, P)$

$B = (B_t)_{t \geq 0}$ std BM

$\mathcal{F}_t^0 = \sigma\{B_s / s \leq t\}$

$\mathcal{F}_t^+ = \bigcap_{s > t} \mathcal{F}_s^0$

$\overline{\mathcal{F}_t^+} = \overline{\mathcal{F}}_t^+$

Fix $T > 0$ and let $(\theta_T B)(t)$

$= B(T+t) - B(T)$

Then
(i) $\theta_T B$ is a std BM
(ii) $\theta_T B$ is independent of $\overline{\mathcal{F}}_T^+ = \overline{\mathcal{F}_T^+}$
hence indep of $\mathcal{F}_T^+$, $\overline{\mathcal{F}}_T$, $\mathcal{F}_T^0$

<u>Proof</u>:— Need only to prove
Let $W = \theta_T B$. First we p.t W is indep of $\mathcal{F}_T^+$
Fix any $0 < t_1 < t_2 < \dots < t_n$
Take $0 < \varepsilon < t_1$
We know that $(W(t_1), \dots W(t_n))$ is indep of $\mathcal{F}_{T+\varepsilon}^0$ (Markov property I)
$\Rightarrow (W(t_1), \dots W(t_n))$ is indep of $\mathcal{F}_T^+ \subseteq \mathcal{F}_{T+\varepsilon}^0$

For $t_1, \dots t_n > 0$
$W(0) = 0$ is also indep of $\mathcal{F}_{T+t}^0$
$\Rightarrow W$ is indep of $\mathcal{F}_T^+$

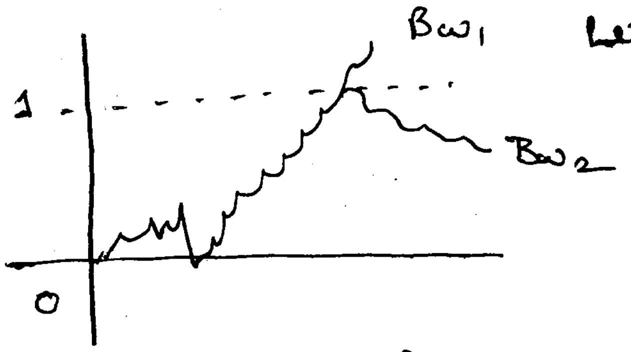
$A \in N \Rightarrow P(A) = 0$
$\Rightarrow A$ is indep of W
Hence $\overline{\mathcal{F}_T^+} = \sigma\{\mathcal{F}_T^+ \cup N\}$ is independent of W
(complete the argument)

Reasons for enlarging filtrations

1) MP for larger filtration is a stronger statement.

2) To get more stopping times

eg:



Let $\tau = \inf\{B_t > 1\}$

$B^{(s)}_{\omega_1} = B^{(s)}_{\omega_2} \quad \forall s \le t_0$

But $\tau(\omega_1) = t_0 < \tau(\omega_2)$

$\Rightarrow \{\tau \le t_0\} \notin \mathcal{F}^{\circ}_{t_0}$

But $\mathcal{F}^{+}_{t_0}$ can distinguish between the two paths.

$\mathcal{G}$: Applications of Markov property:—

Blumenthal's 0-1 law :— $(\Omega, \mathcal{F}, P)$, B = std BM

$\mathcal{F}^{\circ}_t$, $\mathcal{F}^{+}_t$ etc as before. Then $\forall A \in \mathcal{F}^{+}_0$

$P(A) = 0$ or 1

Proof :— If $A \in \mathcal{F}^{+}_0$ by MP

$\sigma\{(B_t)_{t \ge 0}\}$ is independent of A.

But $A \in \mathcal{F}^{+}_0 \subseteq \sigma\{B_t \mid t \ge 0\}$

$\Rightarrow$ A is indep of A

$\Rightarrow \quad P(A) = P(A \cap A) = P(A)P(A)$

$\Rightarrow \quad P(A) = 0$ or 1

Corollary :— [Kolmogorov's 0-1 Law]

Let $\mathcal{T} = \bigcap_{t>0} \sigma\{B_s \mid s \geq t\}$

tail σ-field.

Then $\forall A \in \mathcal{T}$, $P(A) = 0$ or 1

<u>Oscillations of BM near $t = 0$</u>

Let $\tau_+ = \inf\{t \geq 0 \mid B_t > 0\}$

$\tau_- = \inf\{t \geq 0 \mid B_t < 0\}$

Then $P\{\tau_+ = 0, \tau_- = 0\} = 1$

<u>Proof</u> :— ~~$P\{\tau_+ = 0, \tau_- = 0\} = 1$~~

$\{\tau_+ = 0\} = \{\omega \mid \exists\, t_1(\omega) > t_2(\omega) > \cdots \to 0$ s.t $B(t_i(\omega)) > 0\}$

$\in \mathcal{F}_0^+$

Similarly $\{\tau_- = 0\} \in \mathcal{F}_0^-$

Hence by Blumenthal's law

$P\{\tau_+ = 0\}$, $P\{\tau_- = 0\}$ are 0 or 1

Since $B \stackrel{d}{=} -B$, $P\{\tau_+ = 0\} = P\{\tau_- = 0\}$

Thus $P\{\tau_+ = 0\} = P\{\tau_- = 0\} = 0$ or 1

$(\{\tau_+ = 0\} \cup \{\tau_- = 0\})^c \subseteq \bigcap_{\varepsilon > 0} \{B_s = 0 \ \forall s \leq \varepsilon\}$

RHS has prob 0

$\Rightarrow P[\{\tau_+ = 0\} \cup \{\tau_- = 0\}] = 1$

$\Rightarrow P\{\tau_+ = 0\} = P\{\tau_- = 0\} = 1$

<u>Near $t=\infty$.</u> w.p. 1 $\limsup_{t\to\infty} B_t = +\infty$

and $\liminf_{t\to\infty} B_t = -\infty$

($\Rightarrow$ BM hits every pt on the line.)

<u>Proof:</u>— $L^+ = \limsup_{t\to\infty} B_t$ and $L^- = \liminf_{t\to\infty} B_t$

are $\mathcal{T}$-ble.
(tail σ-field).

Hence L^+ and L^- are a.s. constants

By symmetry $L^+ = -L^- = c$ a.s.

for some $0 \le c \le \infty$

If $c \neq \infty$ then $\limsup_t |B_t| < \infty$

$\Rightarrow \sup_t |B_t| < \infty$

But $P\{\sup_t |B_t| < \infty\} = 0$

$\Leftarrow$ for any $M>0$
$P\{\sup_t |B_t| < M\} = 0$

Consider $B(n+1) - B(n)$ iid $N(0,1)$.

w.p. 1 $\exists n$ s.t $B(n+1) - B(n) > 2M$

$\Rightarrow$ $B(n)$ & $B(n+1)$ cannot both be in $[-M, M]$

2/9/09

local-maxima and minima are dense

Theorem: $(\Omega, \mathcal{F}, P)$ B- std BM

let $Max = \{t \mid B(s) \le B(t) \ \forall s \in [t-\delta, t+\delta]$ for some $\delta > 0\}$.

$Min = \{t \mid B(s) \ge B(t) \ \forall s \in [t-\delta, t+\delta]\}$. for some $\delta > 0$.

Then $P\{$ Max and Min dense in $[0,\infty)\} = 1$.

Consider $h > 0$ and Consider BM restricted to $[0, h]$.

$\exists \ t_1, t_2, t_3$ s t for some ω; $B(t_1) < 0$, $B(t_2) > 0$ $B(t_3) \le 0$.

Consider $[t_1, t_3]$ then $\exists \ t^*$ s.t $t_1 < t^* < t_3$ st

$B(t_*) > 0$ and is a local max.

Thus $Max \cap [0, h] \neq \phi$ w.p.1

consider $\theta_T(B)$ then $\theta_T(B)$ is a SBM and hence

$\exists$ a local max for $\theta_T(B)$ in $[0, h]$

hence $B(t)$ has a local max in $[t, T+h]$

Thus $P\{Max \cap [t, T+h]\} \neq \phi$ w.p.1 for each $T > 0$ $h > 0$

hence $P\{\bigcap_{\substack{T \in \mathbb{Q}_+ \\ h \in \mathbb{Q}_+}} \{Max \cap [t, T+h]\}\} = 1$.

$\Rightarrow P\{$Max is dense$\} = 1$

Similarly Min is dense w.p.1.

$\rightarrow$ Does nowhere differentiability imply that max & min are dense

Exercise *

Fix t_0, show that $P\{t_0 \in \text{Max}\} = 0$.

Remark: Let $\text{Incr} = \left\{ t \;\middle|\; \begin{array}{l} \text{for some } \delta > 0 . \\ B(s) \geq B(t) \;\; \forall s \in [t, t+\delta] \\ B(s) \leq B(t) \;\; \forall s \in [t-\delta, t] . \end{array} \right.$

Sec 10

stopping times and strong Markov property:

$\theta_T(B)$ is a SBM

what about taking random times?

~~take~~ It needn't always be a SBM.

eg take $T = \operatorname{argmax}_{0 \leq t \leq 1} B_t$

then $\theta_T(B)$ is not a BM

since for some $\epsilon > 0$ time on $[0, \epsilon]$ $\theta_T(B) \leq 0$.

Eg 2: $T = \max\{t \leq 1 \mid B_t = 0\}$.

$\theta_T(B)$ is not SBM.

If the random time can't foresee into the future then the BM stopped using that stopping time will be a BM.

---

ex. Let $T = \operatorname{argmax}_{0 \leq t \leq 1} B_t$

Prove $\{T \leq 1/2\} \notin \mathcal{F}^0_{1/2}$.

(2) $T = \inf\{s \mid B_s \geq 1\}$. ← is a stopping time.

Prop: $(\Omega, \mathcal{F}, P)$ B - std BM $\mathcal{F}_t^0 = \sigma\{B_s \mid s \le t\}$.

$\mathcal{F}_t^+ = \bigcap_{s>t} \mathcal{F}_s^0$ $\quad \overline{\mathcal{F}}_t^+ = \overline{\mathcal{F}_t^+} = \sigma\{\mathcal{F}_t^+ \cup \mathcal{N}\}$.

1) let A be an open set in $\mathbb{R}^d$ and
let $T_A = \inf\{t : B_t \in A\}$ $t > 0$.
Then T_A is an $\mathcal{F}^+$ stopping time (needn't be $\mathcal{F}_t^0$)

2) let A be a closed set in $\mathbb{R}^d$.
$T_A = \inf\{t \ge 0 \mid B_t \in A\}$ then T_A is a $\mathcal{F}^+$ stopping time

Proof: $T_A \le t \iff$ for any $s > t$ $\exists u < s$ s.t $B_u \in A$.
$\iff$ for any $s > t$ $\exists u < s$ $u \in \mathbb{Q}$ s.t $B_u \in A$.
(A is open & B is cont)

$$\{T_A \le t\} = \bigcap_{s>t}\left(\bigcup_{\substack{u<s \\ u\in\mathbb{Q}}} \{B_u \in A\}\right) \qquad (B_u \in A) \in \mathcal{F}_u \subseteq \mathcal{F}_s.$$

check

$\bigcup_{\substack{u<s \\ u\in\mathbb{Q}}} \{B_u \in A\} \in \mathcal{F}_s.$

$\limsup_{s\to\infty} \left\{\bigcup_{\substack{u<s \\ u\in\mathbb{Q}}} \{B_u \in A\}\right\} \in \mathcal{F}_s.$

eg of a random time s.t shifting by the random tim[e]
is not a BM.

let $T = \inf\{t : B_t \geq 1\} - 1$

$W_t = B(T+t) - B(T)$

$W_1 = B(T+1) - B(T)$

≥ 0

$B(T+1) = 1$

$0 \leq B(T) \leq 1$

Hence not a BM.

<u>sec 9</u> stopping times

$(\Omega, \mathcal{F}, P)$ B - std d-dim BM.

$\mathcal{F}_t^0 = \sigma\{B_s \mid s \leq t\}$ $\mathcal{F}_0^+$ and $\overline{\mathcal{F}_0^+}$ as before.

<u>Prop</u>: Let A be an open set in $\mathbb{R}^d$.

(a) let $\tau_A = \inf\{t \geq 0 \mid B_t \in A\}$ Then τ_A is an $\overline{\mathcal{F}_0^+}$

(b) Let A be a closed set in $\mathbb{R}^d$

Then $\tau_A = \inf\{t \geq 0 \mid B_t \in A\}$ Then τ_A is an $\overline{\mathcal{F}_0^+}$

Proof: a) $\tau_A(\omega) \leq t \Leftrightarrow \forall s > t,\ \omega \in \bigcup_{u<s} \{\omega \mid B_u(\omega) \in A\}$

$$\Leftrightarrow \forall s > t \begin{cases} \omega \in \underbrace{\bigcup_{\substack{u<s \\ u \in \mathbb{Q}}} \{\omega \mid B_u(\omega) \in A\}}_{\to \mathcal{F}} \\ \text{or } B_\omega(\cdot) \text{ is discontinu[ous]} \end{cases}$$

Thus $\{\tau_A \leq t\} = \left[\limsup_{\substack{s \downarrow t \\ s \in \mathbb{Q}}} \bigcup_{\substack{u<s \\ u \in \mathbb{Q}}} \{B_u \in A\}\right]$

Hence $\{\tau_A \leq t\} \in \overline{\mathcal{F}_t^+}$ $\quad \overset{\cap}{\mathcal{F}_t^+}$

(b) let A be a closed set.

let $A_n = \{x \in \mathbb{R}^d : d(x,A) < \frac{1}{n}\}$ open sets in $\mathbb{R}^d$.

Then $\bigcap_n A_n = A$.

$$\tau_A(\omega) \le t \iff \begin{cases} \tau_{A_n}(\omega) \le t & \forall n \\ \text{or} & \\ B_\omega(\cdot) \text{ is discontinuous.} & \end{cases}$$

Hence $\{\tau_A \le t\} = \left[\bigcap_{n=1}^{\infty} \{\tau_{A_n} \le t\}\right] \cup N$ where N is a Null set.

$\mathcal{F}_t^+$ $\forall A_n$ by (a)

$\therefore \{\tau_{A_n} \le t\} \in \mathcal{F}_t^+$

Defining the sigma algebra generated till some random time τ.

Let τ be ~~most~~ an $\mathcal{F}_t$ stopping time

Define $\mathcal{F}_\tau = \{A \in \mathcal{F} \mid A \cap \{\tau \le t\} \in \mathcal{F}_t\}$ $\forall t > 0$.

Ex: check that $\mathcal{F}_\tau$ is a ~~sto~~ σ-field

Ex*: Prove that given stopping times τ_1 and τ_2

$\tau_1 + \tau_2$, $\tau_1 \wedge \tau_2$, $\tau_1 \wedge \tau_2$ are stopping times

and $\tau_1 \tau_2$ $\tau_1 - \tau_2$ are not stopping times.

Find the corresponding σ-fields interms of $\mathcal{F}_{\tau_1}$ and $\mathcal{F}_{\tau_2}$.

example: (i) τ is measurable w.r.t $\mathcal{F}_\tau$.

(ii) Let B_t be a d dim BM on $(\Omega, \mathcal{F}, P)$ then

B_τ is $\mathcal{F}_\tau$ measurable, where τ is a $\mathcal{F}_t^0 / \mathcal{F}_t^+ / \overline{\mathcal{F}}_t^+$ stopping time.

Prop: $(\Omega, \mathcal{F}, P)$, B std d-dim BM.

Let $\mathcal{F}_\cdot, \mathcal{F}_\cdot^+, \overline{\mathcal{F}_\cdot^+}$ as before. Let τ be an $\overline{\mathcal{F}_\cdot^+}$ stopping time

then (i) $W_t = B(\tau+t) - B(\tau)$ $t \geq 0$ is a std d-dim BM.

(ii) W is independent of $\overline{\mathcal{F}_\tau^+}$.

$$\overline{\mathcal{F}_\tau^+} = \{ A \in \mathcal{F} \mid A \cap \{\tau \leq t\} \in \overline{\mathcal{F}_t^+} \ \forall t \geq 0 \}.$$

Proof: Case I Suppose $\exists$ $0 \leq T_1 \leq T_2 < \dots$ (non random)

such that $\sum_k P\{\tau = T_k\} = 1$. (i.e τ takes discrete values)

Now let $A \in \overline{\mathcal{F}_t^+}$ and fix $0 \leq t_1 < t_2 < \dots < t_n$, $\mathcal{C} \in \mathcal{B}_{(\mathbb{R}^d)^n}$

$$P\{(W_{t_1} \dots W_{t_n}) \in \mathcal{C} \text{ and } A\} = \sum_k P\{(W_{t_1} \dots W_{t_n}) \in \mathcal{C} \text{ and } A \text{ and } \{\tau$$

Now $\{A \text{ and } \{\tau = T_k\}\} \in \mathcal{F}_{T_k}$ for each k.

The first term

$$P\{(B_{T_1+t_1} - B_{T_1}, B_{T_1+t_2} - B_{T_1} \dots B_{T_1+t_n} - B_{T_1}) \in \mathcal{C} \text{ and } A \cap \{T=1\}\}$$

By Markov property

$$= P\{(B_{t_1} \dots B_{t_n}) \in \mathcal{C}\} \, P\{A \cap \{T=1\}\}$$

From (*) we have

$$= P\{(B_{t_1} \dots B_{t_n}) \in \mathcal{C}\} \sum_k P(A \cap \{T=k\})$$

$$= P\{(B_{t_1} \dots B_{t_n}) \in \mathcal{C}\} \, P(A).$$

$$\cancel{= P\{(W_{t_1} \dots W_{t_n}) \in \mathcal{C}\} P(A)}.$$

If we take $A = \Omega$

Then we have $P\{(W_{t_1} \dots W_{t_n}) \in \mathcal{C}\} = P\{(B_{t_1} \dots B_{t_n}) \in \mathcal{C}\}$

hence all finite dim distributions are same

thereby proving W_t is std BM, and hence also independent w.r. $\overline{\mathcal{F}^+}$

Case II

general stopping time τ (assume $P(\tau<\infty)=1$)

$\frac{k}{2^n}$ $\quad$ $\frac{k+1}{2^n}$

let $\tau_n = \frac{k+1}{2^n}$ if $\frac{k}{2^n} < \tau \le \frac{k+1}{2^n}$

Then τ_n are stopping times and

Then $\tau_1 \geq \tau_2 \geq \cdots \to \tau$

Then we have $\mathcal{F}^+_{\tau_n} \supseteq \mathcal{F}^+_\tau \ni A$

By case I, for any $0 \le t_1 \cdots < t_m$

$(B(\tau_n+t_1) - B(\tau_n) \;\cdots\; B(\tau_n+t_m) - B(\tau_n))$ is indep of A.

As $n\to\infty$ $\xrightarrow{as} (B(\tau+t_1)-B(\tau), \ldots B(\tau+t_m)-B(\tau))$

$= \text{w.r.t.}\ (W_{t_1} \cdots W_{t_m})$

Hence $(W_{t_1} \cdots W_{t_m})$ indep of A

Prove if $X_1 \cdots X_n$ as $X_n \to X$

if X_i are indep A then X indep of A.

---

last time of exit from a set A is not a stopping time

$E(\sum_1^{\tau} X_n) \; E(X_1 + \cdots + X_\tau)$

$= E(\tau)\, E(X_1)$

zero set $\to$ w.p.1 closed and unbounded.

10/9/09.

Notation: $(\Omega, \mathcal{F}, P)$ be a Prob space, B a d-dim std BM, $\mathcal{F}_t$ (or $\overline{\mathcal{F}_t^+}$) a filtration, then $x+B$ = BM started at x.

Let P_x ~~to~~ denote the distribution of $x+B$ — a probability measure on $C([0,\infty), \mathbb{R}^d)$ and $E_x = \int \cdot \, dP_x$.

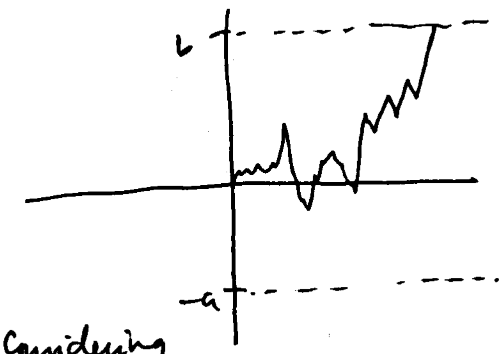
<u>SMP</u>: τ - an $\mathcal{F}_t$ stopping time

Let $(\theta_\tau B)(t) = B(\tau+t) \quad t \geq 0.$

$$\theta_\tau(B) = W + B_\tau.$$

W is ind of $\mathcal{F}_t \equiv$ Cond distribution of $\theta_\tau(B)$ given $\mathcal{F}_t$ is P_{B_τ}.

Gamblers Ruin: $(d=1)$



Let $T_x = \inf\{t \mid B_t = x\}$.

We want to find $P_0\{T_{-a} < T_b\}$

It is easier to solve the problem by considering an arbitrary starting point x instead of 0.

Let $\phi(x) = P_x\{T_1 < T_0\}$.

$\phi(0) = P_0\{T_1 < T_0\} = 0$

$\phi(1) = P_1\{T_1 < T_0\} = 1$.

Let δ be st $(x-\delta, x+\delta) \subset [0,1]$ for $x \in [0,1]$.

<u>Main obs</u>: ~~If~~ $0 \leq x-\delta < x < x+\delta \leq 1$

Then $\phi(x) = \frac{1}{2}\phi(x-\delta) + \frac{1}{2}\phi(x+\delta)$.

$\phi(x)$ Proof: let $\tau = \tau_{x+\delta} \wedge \tau_{x-\delta}$.

$$\phi(x) = P_x\{\tau_1 < \tau_0\}$$

$$= E_x\{P_x\{\tau_1^B < \tau_0^B \mid \mathcal{F}_\tau\}]$$

$$= E_x[P_x\{B_\tau + W \text{ hits } 1 \text{ before } 0 \mid \mathcal{F}_\tau\}] \quad W \to BM \text{ in}$$

$$= E_x[P_{B_\tau}\{\tau_1 < \tau_0\}]$$

$$= E_x[\phi(B_\tau)].$$

$$\phi(0)=0 \quad \phi(1)=1$$

$$\phi(\tfrac{1}{2}) = \tfrac{1}{2}.$$

$$\phi(\tfrac{1}{4}) = \tfrac{1}{2}\phi(0) + \tfrac{1}{2}\phi(\tfrac{1}{2})$$

$$\phi(\tfrac{3}{4})$$

$$\vdots$$

$$\phi(\tfrac{k}{2^n}) = \tfrac{k}{2^n}. \quad \forall n \geq 1 \quad k \leq 2^n.$$

To show $\phi(x) = x \quad \forall x \in [0,1]$ → show ϕ is cont or → show ϕ is non dec.

ϕ is non dec. Let $0 \leq y < x \leq 1$.

$$\phi(y) = P_y\{\tau_1 < \tau_0\}$$

$$= P_y\{\tau_1 < \tau_0, \tau_x < \tau_0\}.$$

$$= E_y\{P_y\{\tau_1 < \tau_0\} \; \tau_x < \tau_0 \mid \mathcal{F}_{\tau_x}\}]$$

$$= E_y[I\{\tau_x < \tau_0\} \cdot P_x\{\tau_1 < \tau_0\}].$$

Ex: $\{T_b < T_{-a}\}$ then $= \frac{a}{a+b}$.

$$B(T_a \wedge T_b) = \begin{cases} -a & \text{wp } \frac{b}{a+b} \\ b & \text{wp } \frac{a}{a+b}. \end{cases}$$

$d\mathcal{F}_t$.

<u>Sec 12</u>:

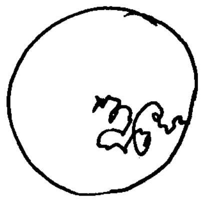
Let $D \subseteq \mathbb{R}^d$ be a bdd open set.

∂D = boundary of D.

$$\left.\begin{aligned} \text{Let } \tau &= \inf\{t : B_t \in \partial D\} \\ &= \inf\{t : B_t \in D^c\}. \end{aligned}\right\} \text{ stopping time since } \partial D \text{ is closed.}$$

Consider the same problem for ~~a circle~~ ~~on~~ the unit disc in $\mathbb{R}^2$.

Then B_τ is the unif dist on S^1.



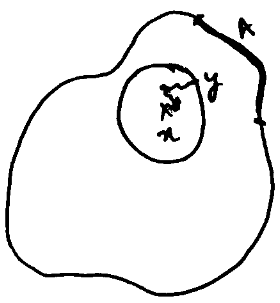
$B_\tau \sim \text{unif } S^{d-1}$ (in d-dim).

By Rotation symmetry (multiplication by an orthogonal matrix).

Consider the general case:

Let A be a Borel set of ∂D.

(consider an arc for simplicity)



Let $A \subset \partial D$

Let $\phi(x) = P_x\{B_\tau \in A\}$.

Suppose $\delta > 0$ s.t $\overline{B(x;\delta)} \subset D$.

Then let $\tilde{\tau} = \inf\{t \mid \|B_t - x\| = \delta\}$.

$$\begin{aligned} \phi(x) &= P_x\{B_\tau \in A\}. \\ &= E_x\left[P_x\{B_\tau \in A \mid \mathcal{F}_{\tilde{\tau}}\}\right] \\ &= E_x\left[P_{B_{\tilde{\tau}}}\{\tilde{B}_\tau \in A\}\right] \quad \tilde{B} \text{ is std BM indep of } \mathcal{F}_{\tilde{\tau}}. \\ &= E_x\left[\phi(B_{\tilde{\tau}})\right]. \quad \because B_{\tilde{\tau}} \text{ is unif on the disc.} \end{aligned}$$

$$= \int_{S^{d-1}} \phi(x+\delta y)\, d\nu(y)$$

where ν = normalized area of S^{d-1}.

Mean value property: $\begin{cases} \phi(x) = \int_{S^{d-1}} \phi(x+\delta y)\, d\nu(y). \\ \forall x \in D \quad \delta < d(x, \partial D) \end{cases}$

Suppose we can show (i) ϕ is cont in D.

(ii) "ϕ is cont up to the boundary"

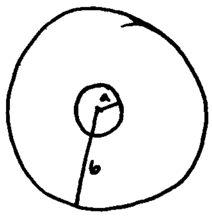
(i) + MVP (or even, (measurability of ϕ) + MVP) $\Rightarrow$ ϕ is C^2 & $\Delta\phi \equiv 0$."

If take also (ii), then ϕ is the unique soln to the Dirichlet prob

$$\begin{cases} \Delta\phi = 0 & \text{in } D \\ \phi|_{\partial D} = I_A . \end{cases}$$

<u>Special case:</u>

$D = \{x \mid a < \|x\| < b\}$.



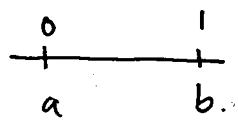
$A = \{\|x\| = b\}$.

$\tau = \inf\{t \mid B_t \in \partial D\}$.

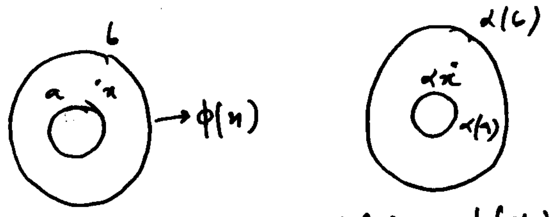
Let $\phi(x) = P_x(\|B_\tau\| = b)$

ϕ has MVP inside D.

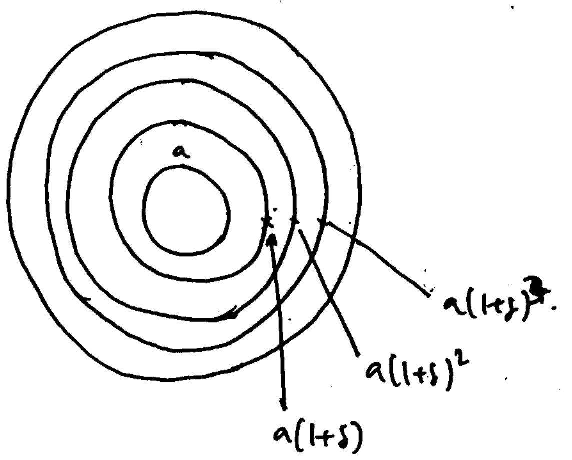
claim: Define $\phi(x) = \begin{cases} 0 & \text{if } \|x\| = a \\ 1 & \text{if } \|x\| = b. \end{cases}$



Scaling:



$\psi(x) = \phi\left(\frac{x}{\alpha}\right)$.



$\varphi \equiv 0$ in D

Let C_r = sphere of radius r.

Start at x where $\|x\| = a(1+\delta)$

$\phi(x) = P_x\{T_b < T_a\}$

Recap: $D \subseteq \mathbb{R}^d$ – open bounded.

$\tau = \inf\{t : B_t \in \partial D\}$ is a stopping time.

We want to find the distribution of B_τ (which depends on the starting pt x).

We know B_τ is an $\mathcal{F}_\tau$ measurable r.v. taking values in ∂D.

Distribution of $B_\tau \equiv E_x\{f(B_\tau)\}$ $\quad f: \partial D \to \mathbb{R}$ bdd Borel measurable.

Let $\phi(x) = E_x[f(B_\tau)]$.

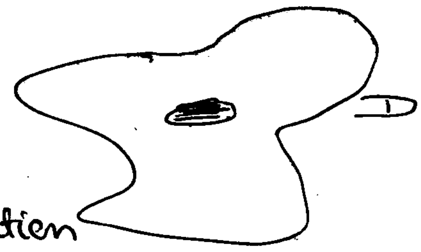
For any f bdd Borel measurable.

(i) ϕ is a Borel measurable fn of x (Exercise)

15-9-2009

Recap:— $D \subseteq \mathbb{R}^d$, open, bounded

$\tau = \inf\{B_t \in \partial D\}$ (stopping time)



We wanted to find the distribution of B_τ (depends on starting pt).

B_τ - $\mathcal{F}_\tau$ measurable r.v. taking values in ∂D.

Distribution of $B_\tau \equiv E_x[f(B_\tau)]$ $f : \partial D \to \mathbb{R}$. bdd Borel measurable fn.

Let $\phi(x) = E_x[f(B_\tau)]$

Last class:- For any f bdd Borel measurable,

(i) ϕ is a Borel meas fn of x (Exer)

(ii) ϕ has MVP i.e

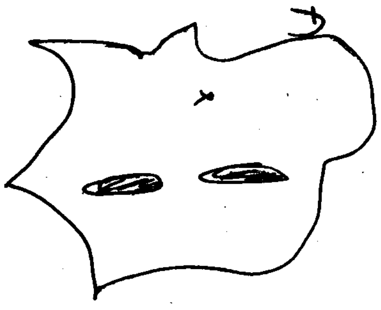
$$\phi(x) = \int_{S^{d-1}} \phi(x+\delta y)\, d\nu(y)$$

where ν = normalized area measure on S^{d-1} $\forall x \in D$, $\delta > 0$ s.t. $\overline{B(x,\delta)} \subseteq D$.

From (i) & (ii), it follows that ϕ is a C^2 and $\Delta\phi(x) = 0$ $\forall x \in D$

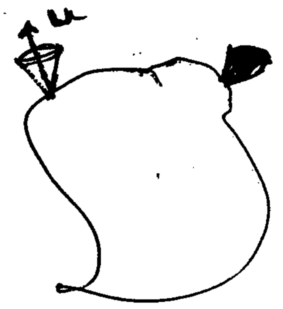
Now fix $f \in C(\partial D)$ (as ∂D is cpt, f is uniformly bdd).

$\phi(x) = E_x[f(B_\tau)]$ for $x \in D$



Q: Behaviour of $\phi(x)$ as $x \to \partial D$

Poincaré's cone condition: D as before, $p \in \partial D$.



We say that PCC is satisfied at p if $\exists$ a cone C with angle $\alpha > 0$ and radius $r > 0$ such that p is the vertex of C and $C \cap D = \phi$.

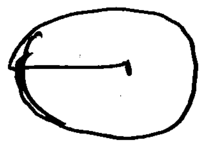
[a cone :- $\exists\, u \in \mathbb{R}^d$, $\|u\| = 1$ s.t

$C = p + \{x \in \mathbb{R}^d \mid \langle x, u\rangle \geq \cos(\alpha/2)\,\|x\|$ & $\|x\| \leq r\}$]

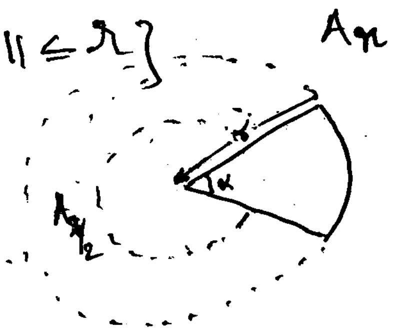
Theorem: If D satisfies PCC at $p \in \partial D$, (with some $\alpha > 0$, $r > 0$) then $\phi(x) \to f(p)$ as $x \to p$, $x \in D$

(conditions as before, $D \to$ bd open, $f \in C(\partial D)$)

Note: PCC not sharpest cond^n

e.g:  (Disc with a slit)

Lemma:- Let $C_r = \{x / \langle x, e\rangle \geq \cos(\alpha/2)\|x\|$, $\|x\| \leq r\}$



Sphere of radius r centred at 0.

Then $\exists\, a < 1$ such that for any $x \in \mathbb{R}^d$, $\|x\| < \frac{r}{2}$, $P_x\{\tau_{A_r} < \tau_{C_r}\} \leq a$ (a depends on α but not on r)

Proof:- If $W_t = \frac{1}{r} B(r^2 t)$ then by scale invariance. W is a std BM

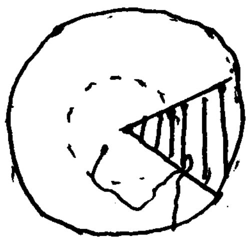
$W_0 = \frac{X}{r}, \|\frac{X}{r}\| \le \frac{1}{2}$, $\quad \tau_{A_1}^W = \frac{\tau_{A_r}^B}{r^2}$

$$\tau_{C_1}^W = \frac{1}{r^2}\tau_{C_r}^B$$

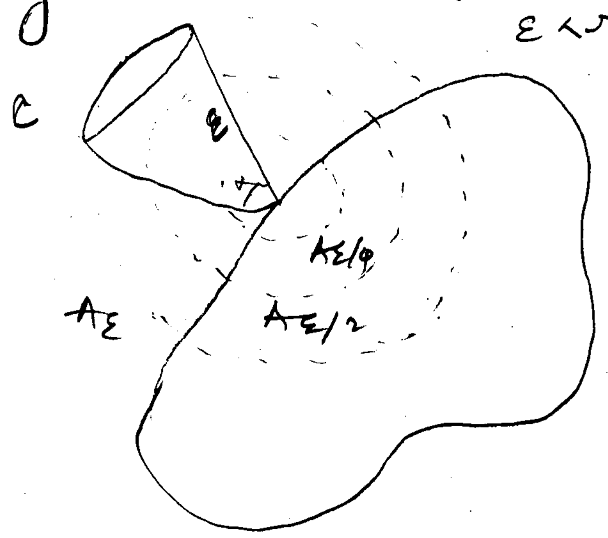
Hence $\tau_{A_1}^W < \tau_{C_1}^W \Longleftrightarrow \tau_{A_r}^B < \tau_{C_r}^B$.

$\Rightarrow$ Enough to consider $r = 1$

Exe* : for $r = 1$



Proof:— (Theorem) $p \in \partial D$ satisfies PCC say with a cone of radius $r > 0$ and angle $\alpha > 0$.



Fix $0 < \varepsilon < r$.

If $\|x - p\| < \frac{\varepsilon}{2^k}$, $x \in D$

then $P_x\{|B_\tau - p| > \epsilon\} < a^k$

Let A_ϵ = sphere of radius ϵ centered at p

Reason :

$P_x\{\|B_\tau - p\| > \varepsilon\}$

$\leq P_x\{\tau_{A_\varepsilon} < \tau_c\}$

$\overset{SMP}{=} P_x\{\tau_{A(\epsilon/2)} < \tau_c\} \; E_x\left[P_{B_{\tau_{A(\epsilon/2)}}}\{\tau_{A_\epsilon} < \tau_c\}\right]$

$\leq P_x\{\tau_{A_{\varepsilon/2}} < \tau_c] \cdot a$

$\leq a^k \cdot P_x[\tau_{A_{\varepsilon/4}} < \tau_c\}$

$\ldots \leq a^{k-1}$

thus $P_x\{\|B_t - p\| > \varepsilon\} \leq \cancel{\frac{1}{\|x-p\|^b} \text{ for}}\; a^{k-1}$

~~some $b>0$~~ if $\|x - p\| < \frac{\varepsilon}{2k}$

$|\phi(x) - f(p)| = |E_x f(B_\tau) - f(p)|$

$\leq E_x[|f(B_t) - B_p)| \, 1(\|B_\tau - p\| < \epsilon)]$

$+ E_x[|f(B_t) - f(p)| \, 1(\|B_t - p\| \geq \varepsilon)]$

$\leq \sup\limits_{\substack{q \in \partial D \\ \|q-p\| < \varepsilon}} \|f(q) - f(p)\| + 2\|f\|_{\sup} P_x\{|B_t - p| > \varepsilon\}$

$\Rightarrow \phi(x) \to f(p)$ as $x \to p$.

<u>Note</u>: we use only the condition that f is cts at p and is bounded & Borel m'ble.

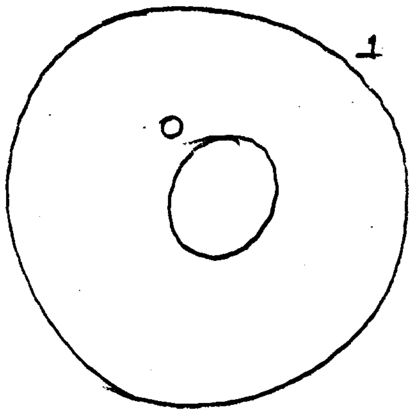
<u>Corollary</u>:— If $D \subseteq \mathbb{R}^d$ bdd open satisfies PCC at every bdry pt. Then given $f \in C(\partial D)$, set $\phi(x) = \begin{cases} E_x[f(B_\tau)], & x \in D, \ \tau = \inf\{t / B_t \in \partial D\} \\ f(x), & x \in \partial D \end{cases}$

Then ϕ is the unique soln to the Dirichlet problem.

$u \in C(\bar{D})$, $\Delta u = 0$ in D, $u|_{\partial D} = f$

<u>Special case</u>:— $D = \{x \in \mathbb{R}^d / a < \|x\| < b\}$

$0 < a < b < \infty$. Then let



$f(p) = \begin{cases} 1 & \text{if } \|p\| = b \\ 0 & \text{if } \|p\| = a \end{cases}$

So $f \in C(\partial D)$

D satisfies PCC at all $p \in \partial D$

Now $\phi(x) = E_x[f(B_\tau)]$

$= P_x\{\tau_b < \tau_a\}$

τ_b = hitting time for bS^{d-1}

By the corollary, ϕ is the unique harmonic fn on D, cts on $\bar{D}$. are equal to $\begin{matrix} 1 \text{ on } \|p\| = b \\ 0 \text{ on } \|p\| = a \end{matrix}$

ϕ is clearly radial. Write $\phi(r)$ $a \le r \le b$

$\phi(a) = 0$, $\phi(b) = 1$

$$\left(\frac{d^2}{dr^2} + \frac{d-1}{r}\frac{d}{dr}\right)\phi(r) = 0$$

$$\phi''(r) = -\frac{d-1}{r}\phi'(r)$$

$$\Rightarrow \phi'(r) = \frac{c}{r^{d-1}}$$

$$\Rightarrow \phi(r) = \begin{cases} \dfrac{c'}{r^{d-2}} + c'' & d \ge 3 \\ c' \log r + c'' \end{cases}$$

Using $\phi(a) = 0$, $\phi(b) = 1$ given

$$\phi(x) = \frac{\|x\|^{-(d-2)} - a^{-(d-2)}}{b^{-(d-2)} - a^{-(d-2)}}, \quad d \ge 3$$

$$\phi(x) = \frac{\log \|x\| - \log a}{\log b - \log a}, \quad d = 2.$$

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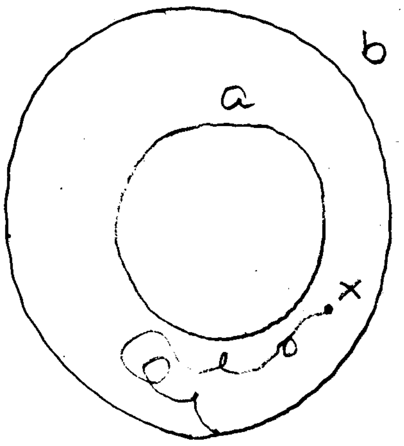


$\mathbb{R}^d$, B - d-dim B.M

$a \le \|x\| \le b$

$$P_x\{\tau_b < \tau_a\} = \begin{cases} \dfrac{\|x\|^{-d+2} - a^{-d+2}}{b^{-d+2} - a^{-d+2}}, & d \geq 3 \\[2ex] \dfrac{\log\|x\| - \log a}{\log b - \log a}, & d = 2 \end{cases}$$

consequences:—

1) $d \ge 2$. Fix b and let $a \downarrow 0$. Fix $x \neq 0$, $\|x\| < b$ } Then $P_x\{\tau_b < \tau_a\} \to 1$

In particular, if $\tau_0 = \inf\{t \mid B_t = 0\}$

Then $P_x\{\tau_0 > \tau_b\} \geq P_x\{\tau_a > \tau_b\}$, $\forall\, a > 0$, $a < \|x\|$

$\to 1$ as $a \downarrow 0$

Thus $P_x\{\tau_0 > \tau_b\} = 1$ for any $x \neq 0$

and any $b > \|x\|$.

Hence $\tau_0 = \infty$ w.p.1.

In $d \geq 2$ B.M does not hit points (Point transience)

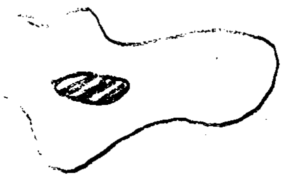
Exer: Fix $a > 0$ and let $b \uparrow \infty$ and show that

1) $d = 2$: a) w.p.1 B hits every neighbourhood of every point. (b) $\forall$ open disk $B(y,\delta)$, $\exists\, t_1 < t_2 < \cdots \to \infty$ (unend times) s.t. $B(t_i) \in B(y,\delta)$.

2) $d \geq 3$: w.p.1. $\|B_t\| \to \infty$ as $t \to \infty$

Property (1 b) is called as neighbourhood recurrent and (2) is called as neighbourhood transience

Loose ends :—



(1) D - bdd open in $\mathbb{R}^d$

Dirichlet problem :— Given $f : \partial D \to \mathbb{R}$ f continuous find $u : \overline{D} \to \mathbb{R}$

s.t (a) u is cts on $\overline{D}$

(b) u is harmonic in D $(\Delta u \equiv 0)$

(c) $u|_{\partial D} = f$

We have shown that D satisfies PCC then

$$u(x) = \begin{cases} E_x[f(B_\tau)] , & x \in D \\ f(x) , & x \in \partial D \end{cases}$$ is the solution to D.P

where

$\tau = \inf \{ t \geq 0 \mid B_t \in \partial D \}$

If $f : \partial D \to \mathbb{R}$ is bd Borel, not continuous

one can ask for

(b) and (c) but (a) is not meaningful

In some sense, $u(x) = E_x[f(B_\tau)]$ is always the solution to the DP (whatever sense of bdry condns we take)

2) For $x \in D$, let $\mu_x(A) = P_x\{B_\tau \in A\}$

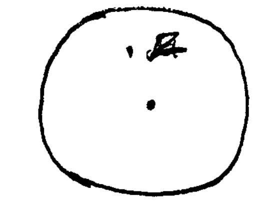
|| Note $p \in \partial D$, $(d=2)$ is a good bdry pt if $\exists$ $K \subseteq \mathbb{R}^2$ connected, $p \in K$, & $K \cap D = \phi$. ||

$\to \mu_x$ = distribution of B_τ - a p.m supported on ∂D

μ_x is called the Harmonic measure of

∂D as seen from x. (*)

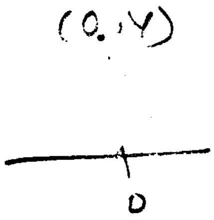
Eg : $d = 2$, $D = \{ |z| < 1 \}$



$$\frac{d\mu_z(\theta)}{d\theta} =: P(z, \theta) \quad \text{Poisson kernel of unit disc.}$$

$$= \frac{1}{2\pi} \cdot \frac{1 - |z|^2}{|z - e^{i\theta}|^2}$$

Exercise* Let $D = \{ (x, y) / y > 0 \}$

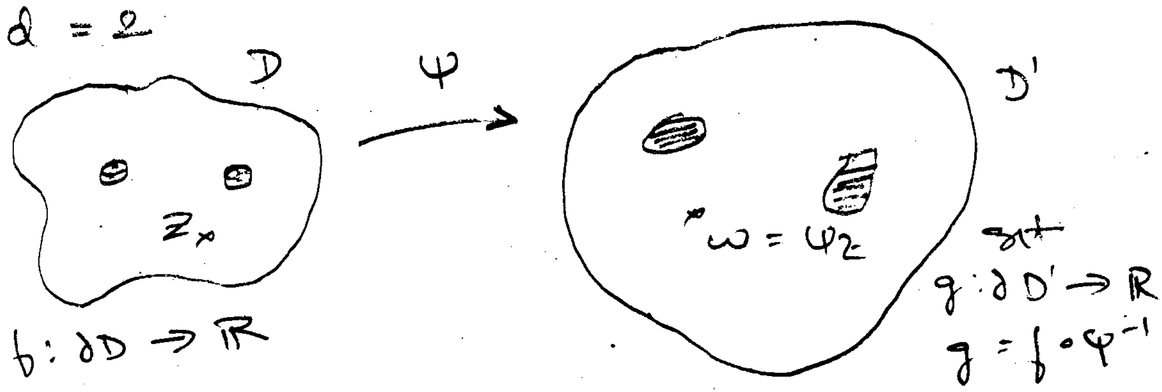


Find $\mu_{(0,y)}$

At least show that $\mu_{(0,y)}$ is Cauchy parameter c_y.

(*) In terms of μ_x, $u(x) = \int_{\partial D} f(p)\, d\mu_x(p)$

4) $d = 2$



Assume ψ extends to a homeomorphism of $\bar{D}$ to $\bar{D}'$ 1-1 onto, analytic

Then if u is a solution to DP on D, with bd condition f. Then $v := u \circ \psi^{-1}$ is the solution to DP on D' with Bd condition g

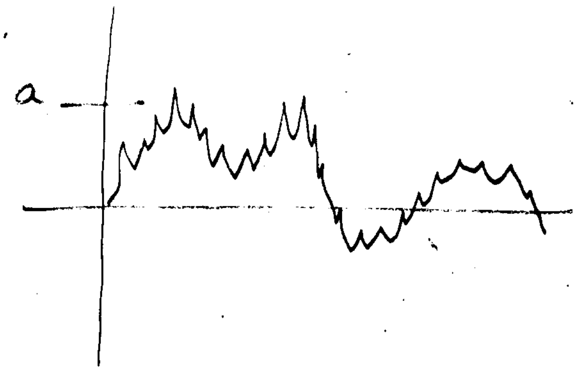
From this it follows that

$$\mu_\omega^{D'} = \mu_z^D \circ \varphi^{-1} \qquad \text{where } \omega = \varphi z.$$

$$\varphi(B_{z_0}) \overset{d}{=} W_{z_0}$$

<u>Sec. 15</u> Reflection principle, Running max, First passage times.

B = 1-dim std B.M.



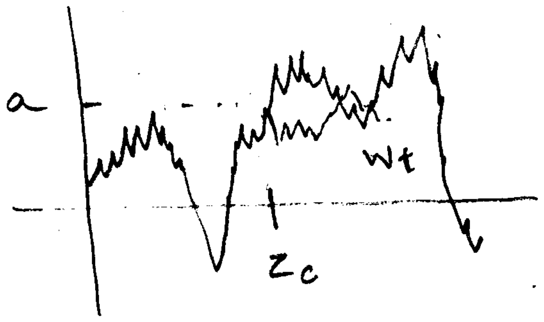
$$\tau_a = \inf\{t \mid B_t = a\}$$

$$M_t = \max_{s \le t} B_s$$

<u>Reflection principle</u>:— B std 1-dim B.M.

$\tau_a = \inf\{t \mid B_t = a\}$

Then let $W_t = \begin{cases} B_t, & t \le \tau_a \\ 2a - B_t, & t > \tau_a \end{cases}$



Then W is a std BM.

Proof:— τ_a is a stopping time. Hence by SMP
$X_t = (B_t - B_{\tau_a})_{t \geq \tau_a}$ is a std 1-dim B.M independent of $\mathcal{F}_{\tau_a}$.

Then $-(B_t - B_{\tau_a})_{t \geq \tau_a}$ is also std 1-dim BM ind of $\mathcal{F}_{\tau_a}$

$Y = (B_t)_{t \leq \tau_a}$ is $\mathcal{F}_{\tau_a}$ measurable.

X, Y : independent
$-X, Y$: — " —
$X, -X$: identical

$\Rightarrow (Y, X) \stackrel{d}{=} (Y, -X)$

B is made by concatenating Y and X

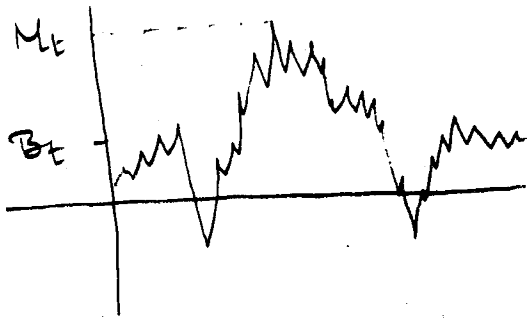
W is — " — — " — Y and $-X$

Hence $B \stackrel{d}{=} W$

Note $(B_{\tau_a} = a)$

Running maximum:—

$M_t = \max_{s \leq t} B_s$

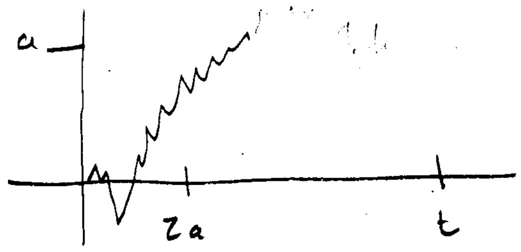


Fix t. Consider (M_t, B_t)

Fix $x \leq a$ $P_0\{M_t \geq a, B_t \leq x\}$

$= P\{\tau_a \leq t, B_t \geq 2a - x\}$

(Using reflection principle)



$= P\{W_t \geq 2a - x\}$

$= P\{\chi > \frac{2a-x}{\sqrt{t}}\}$ where χ is $N(0,1)$

Let f = pdf of (M_t, S_t)

for $a \geq 0$, $-\infty < x \leq a$

$$f(a, x) = \frac{-\partial^2}{\partial a\,\partial x} P\{M_t \geq a, S_t \leq x\}$$

$$= \frac{-\partial^2}{\partial a\,\partial x} P\{W_t > 2a - x\}$$

$$= \frac{\partial}{\partial x}\left[\frac{2e^{-(2a-x)^2/2t}}{\sqrt{2\pi t}}\right]$$

$$= \frac{2\cdot(2a-x)}{\sqrt{2\pi}\;t^{3/2}} \cdot e^{-(2a-x)^2/2t}$$

22-09-2009

Recap :- B : std 1-dim B.M.

$$M_t = \max_{s \le t} B_s$$

Then, $P\{M_t \geq a, B_t \leq x\}$, $-\infty < x \leq a$, $a \geq 0$

$= P\{B_t \geq 2a - x\}$ $\quad (*)$

Applying $-\frac{\partial^2}{\partial a \partial x}$ we get

$$f_{(M_t, B_t)}(a, x) = \frac{2(2a-x)}{\sqrt{2\pi} \cdot t^{3/2}} e^{-(2a-x)^2/2t}, \quad a \geq 0, -\infty < x \leq a$$

Consequences $\qquad$ fixing $t \geq 0$, for $a \geq 0$

1) Distribution of M_t: $P\{M_t \geq a\}$

$= P\{M_t \geq a, B_t \geq a\} + P\{M_t \geq a, B_t \leq a\}$

$= P\{B_t \geq a\} + P\{B_t \geq 2a - a\}$ by $(*)$

$= 2P\{B_t \geq a\}$

$= P\{|B_t| \geq a\}$

Thus $M_t \stackrel{d}{=} |B_t|$

Clearly $M_\cdot \stackrel{d}{\neq} |B_\cdot|$ ($\because$ $M_\cdot$ is weakly increasing)

Exer.* Fix $t > 0$. Then $M_t - B_t \stackrel{d}{=} |B_t|$

Note: $M. - B. \overset{d}{=} |B.|$

2) First passage times: For $a \geq 0$

recall $\tau_a = \inf\{t / B_t = a\}$

Fix $a > 0$. Then $P\{\tau_a \leq t\}$

$= P\{M_t \geq a\}$

$= P\{|B_t| \geq a\}$

$= P\{|\chi| \geq \frac{a}{\sqrt{t}}\}$ $(\chi \sim N(0,1))$ $= \int_{a/\sqrt{t}}^{\infty} 2 \cdot \frac{e^{-u^2/2}}{\sqrt{2\pi}}\, du$

Density of τ_a is

$$\frac{\partial}{\partial t} P\{\tau_a \leq t\} = \frac{2 e^{-a^2/2t}}{\sqrt{2\pi}} \cdot \frac{a}{2t^{3/2}}$$

$$= \frac{a e^{-a^2/2t}}{\sqrt{2\pi}\, t^{3/2}}$$

Note that $E[\tau_a] = \infty$

3) Second look at F.P.T.:—

a) Fix $a > 0$, $b > 0$ Then let

$$W_t = B_{\tau_a + t} - B_{\tau_a}, \quad t \geq 0$$

$$= B_{\tau_a + t} - a$$

By SMP, W is a std B.M.

independent of $\mathcal{F}_{\tau_a} = \sigma\{B_s / s \leq \tau_a\}$

Then $\boxed{\tau_{a+b} = \tau_a + \tau_b^W}$ where $\tau_b^W = \inf\{t / W_t = b\}$

$\tau_b^W \overset{d}{=} \tau_b$

and τ_b^W is independent of τ_a.

(b) $\boxed{\tau_{2a} \overset{d}{=} 4\tau_a}$

Let $W_t = 2B(t/4)$

By scaling invariance, W is a std B.M.

$\tau_{2a}^W = 4\tau_a^B$

$\overset{d}{\|}$

τ_{2a} & hence the result.

From (a) & (b) together, we get

$4\tau_a \overset{d}{=} \tau_{2a} = \tau_{a+a} \overset{d}{=} \tau_a + \tilde{\tau}_a$

where $\tilde{\tau}_a$ is an independent copy of τ_a.

Equivalently $\dfrac{\tau_a + \tilde{\tau}_a}{4} \overset{d}{=} \tau_a \quad (**)$ (This is called as stable ½ distrib.)

Remark (stable 2 distribn) $\dfrac{X + \tilde{X}}{\sqrt{2}} \overset{d}{=} X$, $X, \tilde{X}$ iid $\Rightarrow X \sim N(0, \sigma^2)$ are the only sol'ns.

$\dfrac{X + \tilde{X}}{2} \overset{d}{=} X \Rightarrow X \sim \text{Cauchy}(\alpha)$

(pdf is $\dfrac{\alpha}{\pi(\alpha^2 + x^2)}$ on $\mathbb{R}$).

How to solve for distribution of τ_a?

Let $\psi_a(\lambda) = E[e^{-\lambda \tau_a}]$ $\lambda > 0$

(**) From (a),

$$\psi_{a+b}(\lambda) = \psi_a(\lambda) \cdot \psi_b(\lambda)$$

From (b), $\psi_{2a}(\lambda) = \psi_a(4 \cdot \lambda)$

From these two, we get $\psi_a(4 \cdot \lambda) = (\psi_a(\lambda))^2$

$(\tau_a + \tilde{\tau}_a \overset{d}{=} 4 \cdot \tau_a)$

If $\phi_a(\lambda) = \psi_a(\lambda^2)$

$$= \phi_a(2\sqrt{\lambda}) = (\phi_a(\sqrt{\lambda}))^2$$

$$= \phi_a(2\mu) = (\phi_a(\mu))^2$$

$\Rightarrow \phi_a(\mu) = e^{-\beta_a \mu}$, β_a - a constant

$\Rightarrow \psi_a(\lambda) = e^{-\beta_a \sqrt{\lambda}}$, $\lambda \geq 0$

From $\psi_{a+b}(\lambda) = \psi_a(\lambda) \cdot \psi_b(\lambda)$

We get $e^{-\beta_{a+b}\sqrt{\lambda}} = e^{-\beta_a \sqrt{\lambda}} \cdot e^{-\beta_b \sqrt{\lambda}}$

$$\Rightarrow \beta_{a+b} = \beta_a + \beta_b$$

$$\Rightarrow \beta_a = \gamma a \quad , \gamma \text{ - constant.}$$

Thus $\psi_a(\lambda) = e^{-\gamma a \sqrt{\lambda}}$

γ is as yet undetermined.

Lévy's identity: B: 1-dim B.M.

$M_t = \max_{s \le t} B_s$, $X_t = M_t - B_t$, $Y_t = |B_t|$

Then $X \stackrel{d}{=} Y$ (i.e. $(X_t)_{t \ge 0} \stackrel{d}{=} (Y_t)_{t \ge 0}$)

Proof :— Both X & Y are cts. [illegible]

We need to show that

$(X_{t_1}, \ldots, X_{t_n}) \stackrel{d}{=} (Y_{t_1}, \ldots, Y_{t_n})$

$\forall\ t_1 < \cdots < t_n$, $\forall n$.

$n = 1$ is enough.

For general n, it is enough to show that

$X_{t_1} \stackrel{d}{=} Y_{t_1}$, $X_{t_2} | X_{t_1} = x_1 \stackrel{d}{=} Y_{t_2} | Y_{t_1} = x_1$

$X_{t_n} | X_{t_1} = x_1, \ldots, X_{t_{n-1}} = x_{n-1} \stackrel{d}{=} Y_{t_n} | Y_{t_1} = x_1, \ldots, Y_{t_{n-1}} = x_{n-1}$

[illegible]

$\Leftarrow$ For any $t > s \ge 0$

$X_t | \{X_s = x, \mathcal{F}_s\} \stackrel{d}{=} Y_t | \{Y_s = x, \mathcal{F}_s\}$ & the conditional distribution depends only on x.

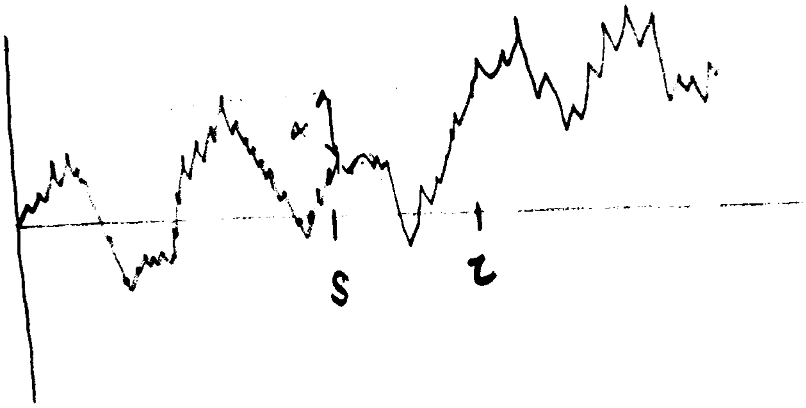
where $\mathcal{F}_s = \sigma\{B_u \mid u \le s\}$

e.g: $(X, Y) \sim f(x, y)$

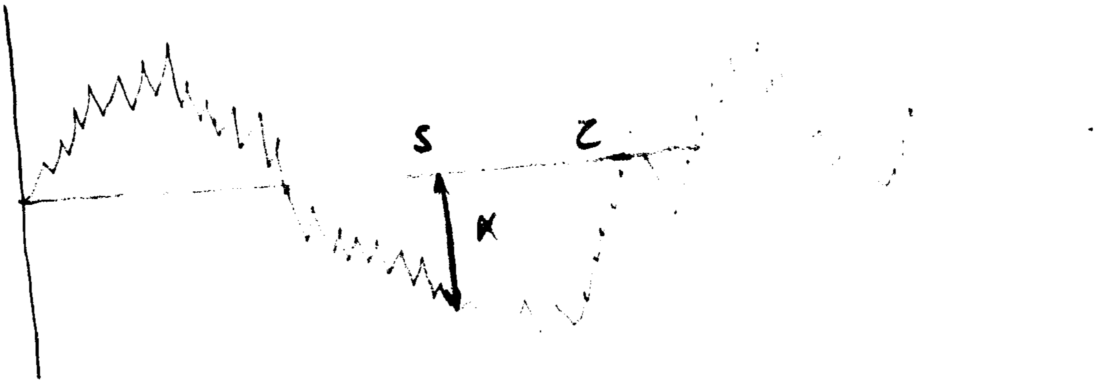
$X|_{Y=y_0}$ is just the pdf $g(x) = \dfrac{f(x, y_0)}{\int f(u, y_0)\,du}$

Actually X & Y have markov property & have same transition probability

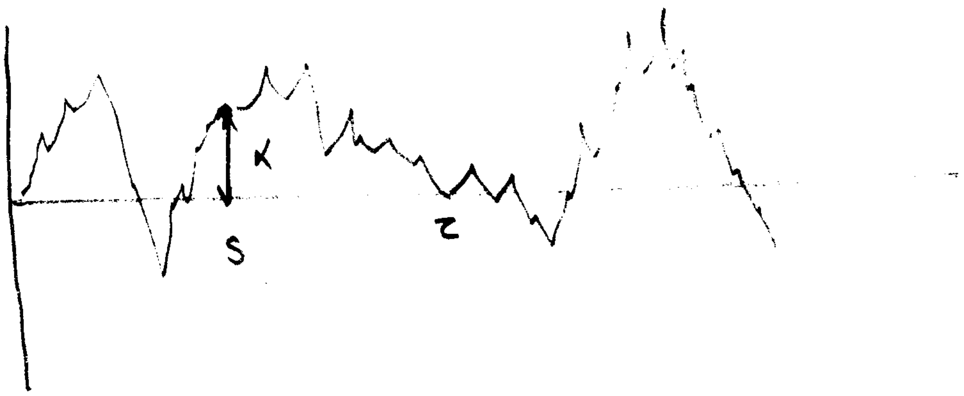
M_s



Case 1



case 2



$(B_u)_{u \le s}$

$(W_u)_{0 \le u \le \tau - s}$ These are independent

$(\tilde{W}_u)_{u \ge 0}$

$W_u = B_{u+s} - B_s$

$\tau = s + \inf\{u / W_u = x\}$

$\tilde{W}_u = B_{\tau+u} - B_\tau$

Y

Case 1 (WLOG)

$(B_u)_{u \le s}$ $(W_u)_{u \le \tau - s}$ $(\tilde{W}_u)_{u \ge 0}$

$W_u = B_{u+s} - B_s$

$\tau = s + \inf\{u / W_u = x\}$ independent

$\tilde{W}_u = B_{\tau+u} - B_\tau$ $\tilde{W}$, W is B.M.

Y

$t > s$

$$B_t = \begin{cases} B_s + W_{t-s} & \tau > t \\ x + \tilde{W}_{t-\tau} & \tau \le t \end{cases}$$

$$M_t = \begin{cases} M_s & \tau > t \\ M_s + \tilde{M}_{t-\tau} & \tau \le t \end{cases}, \quad \tilde{M}_u \text{ - running max for } \tilde{W}$$

$$X_t = M_t - B_t = \begin{cases} \alpha - W_{t-s} & \tau > t \\ \tilde{M}_{t-\tau} \tilde{W}_{t-\tau} & \tau \le t \end{cases}$$

Y

Case 1 for simplicity :—

$$B_t = \begin{cases} B_s + W_{t-s} & \tau > t \\ \quad (\alpha + W_{t-s} < 0) & \\ \tilde{W}_{t-\tau} & \tau \le t \end{cases}$$

$$Y_t = \begin{cases} \alpha - W_{t-s} & t < \tau \\ |\tilde{W}_{t-\tau}| & t \geq \tau \end{cases}$$